ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Fourier sine transform***
- PYQ
- DPP

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or f(p) and defined as

OR
F(s)

$$F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx$$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipx}dp$$

Fourier sine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$ or $f_s(p)$ and defined as

$$F_s\{F(x)\} = f_s(p) = \int_0^\infty F(x) \sin px\, dx$$

Where p is a parameter and $p > 0$

Inverse Fourier sine transform

If $F_s\{F(x)\} = f_s(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier sine transform and defined as

$$F(x) = F_s^{-1}\{f_s(p)\} = \frac{2}{\pi} \int_0^\infty f_s(p) \sin px\, dp$$

Some Important Points and Formulae

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Two Important results

(i) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + c$

(ii) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + c$

Trigonometric Identities

(1) $2 \sin A \cos A = \sin 2A$

(2) $2 \sin A \cos B = \sin (A+B) + \sin (A - B)$

(3) $2 \cos A \sin B = \sin (A+B) - \sin (A - B)$

(4) $2 \cos A \cos B = \cos (A+B) + \cos (A - B)$

(5) $2 \sin A \sin B = \cos (A-B) - \cos (A + B)$

Q.1. *Find the Fourier sine transform of* $F(x) = e^{-3x} + e^{-4x}$

$$F(x) = e^{-3x} + e^{-4x}$$

By Fourier sine transform

$$f_s(p) = \int_0^\infty F(x) \sin px \, dx$$

$$f_s(p) = \int_0^\infty \left(e^{-3x} + e^{-4x}\right) \sin px \, dx$$

$$f_s(p) = \int_0^\infty e^{-3x} \sin px \, dx + \int_0^\infty e^{-4x} \sin px \, dx$$

$$f_s(p) = \left[\frac{e^{-3x}}{(-3)^2+p^2}(-3\sin px - p\cos px)\right]_0^\infty$$

$$+\left[\frac{e^{-4x}}{(-4)^2+p^2}(-4\sin px - p\cos px)\right]_0^\infty$$

$$f(p) = \left[0 - \frac{1}{9+p^2}(0-p)\right] +$$

$$\left[0 - \frac{1}{16+p^2}(0-p)\right]$$

$$f_s(p) = \left[\frac{-1}{9+p^2}(-p)\right] + \left[\frac{-1}{16+p^2}(-p)\right]$$

$$\boxed{f_s(p) = \frac{p}{p^2+9} + \frac{p}{p^2+16}}$$

Q.2 . Find the Fourier sine transform of $e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$.

$$F(x) = e^{-|x|}$$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

For Fourier sine transform

$$x > 0$$

$$\Rightarrow |x| = x$$

$$\Rightarrow F(x) = e^{-x}$$

$$f_s(p) = \int_0^\infty F(x) \sin px\, dx$$

$$= \int_0^\infty e^{-x} \sin px\, dx$$

$$= \left[\frac{e^{-x}}{(-1)^2 + p^2} \left(-\sin px - p \cos px \right) \right]_0^\infty$$

$$= \left[0 - \left(\frac{1}{1+p^2} (-p) \right) \right]$$

$$f_s(p) = -\left(\frac{-p}{1+p^2}\right)$$

$$\boxed{f_s(p) = \frac{p}{p^2+1}}$$

By Inverse Fourier sine Transform

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(p)\sin px\,dp$$

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2+1}\sin px\,dp$$

Put $x = m$

$$F(m) = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2+1}\sin pm\,dp$$

$$\int_0^\infty \frac{p}{p^2+1}\sin pm\,dp = \frac{\pi}{2}F(m)$$

Replace p by x

$$\boxed{\int_0^\infty \frac{x}{x^2+1}\sin mx\,dx = \frac{\pi}{2}e^{-m}}$$

Q.3. Find Fourier sine transform of $\frac{e^{-ax}}{x}$, $a > 0$. Hence find the Fourier sine transform of $\frac{1}{x}$ (GBTU-2011; UPTU-2015)

$$F(x) = \frac{e^{-ax}}{x}$$

$$I = \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx \quad \text{——(1)}$$

By Fourier sine transform

$$f_s(p) = \int_0^\infty F(x) \sin px \, dx$$

$$f_s(p) = \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx$$

Let $f_s(p) = I$

differentiate w.r.t p

$$\frac{dI}{dp} = \int_0^\infty \frac{e^{-ax}}{x} \frac{d}{dp}(\sin px) \, dx$$

$$\frac{dI}{dp} = \int_0^\infty \frac{e^{-ax}}{x} \times x \cos px \, dx$$

$$\frac{dI}{dp} = \int_0^\infty e^{-ax} \cos px \, dx$$

$$\frac{dI}{dp} = \left[\frac{e^{-ax}}{(-a)^2 + p^2} \left(-a \cos px + p \sin px \right) \right]_0^\infty$$

$$\frac{dI}{dp} = \left[0 - \left\{ \frac{1}{a^2 + p^2} (-a) \right\} \right]$$

$$\frac{dI}{dp} = \frac{a}{a^2 + p^2}$$

$$dI = \frac{a}{a^2 + p^2} dp$$

Integrate w.r.t p

$$\int dI = \int \frac{a}{a^2 + p^2} dp$$

$$I = \tan^{-1}\left(\frac{p}{a}\right) + c \quad -(2)$$

$$f_s(p) = \tan^{-1}\left(\frac{p}{a}\right) + c$$

Let initially $p = 0$

Put $p = 0$ in (1)

$$I = 0$$

Put $p = 0$ and $I = 0$ in (2)

$$0 = 0 + c$$

$$c = 0$$

$$\boxed{f_s(p) = \tan^{-1}\left(\frac{p}{a}\right)}$$

$$f_s(p) = \tan^{-1}\left(\frac{p}{a}\right)$$

$$\int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx = \tan^{-1}\left(\frac{p}{a}\right)$$

Put $a = 0$

$$\int_0^\infty \frac{1}{x} \sin px \, dx = \tan^{-1}\left(\frac{p}{0}\right) = \tan^{-1}(\infty)$$

$$\boxed{\int_0^\infty \frac{1}{x} \sin px \, dx = \frac{\pi}{2}}$$

Q.4. *Find $F(x)$ if its Fourier sine transform is e^{-ap}*

$$f_s(p) = e^{-ap}$$

By inverse Fourier sine transform

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(p)\sin px\,dp$$

$$= \frac{2}{\pi}\int_0^\infty e^{-ap}\sin px\,dp$$

$$= \frac{2}{\pi}\left[\frac{e^{-ap}}{(-a)^2+x^2}\left(-a\sin px - x\cos px\right)\right]_0^\infty$$

$$F(x) = \frac{2}{\pi}\left[0 - \left\{\frac{1}{a^2+x^2}(-x)\right\}\right]$$

$$F(x) = \frac{2}{\pi}\left(\frac{x}{a^2+x^2}\right)$$

$$\boxed{F(x) = \frac{2x}{\pi(a^2+x^2)}}$$

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Topic : *Fourier sine transform*

Q.1. *Find Fourier sine transform of* $\frac{1}{x}$

Q. 2. *Find the Fourier sine transform of Block function* $f(t)$ *of height 1 and duration a defined by*

$$f(t) = \begin{cases} sinx, & 0 < x < a \\ 0, & x > a \end{cases}$$

Q.3. *Find* $F_s^{-1}\left(\frac{e^{-ap}}{p}\right)$ *and hence evaluate* $F_s^{-1}\left(\frac{1}{p}\right)$. *(AKTU-2016)*

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