Engineering Mathematics

Unit: Partial Differential Equations (PDE)

Lec-5

Today's Target

- Lagrange's Method [Part-2]
- Univ. Questions
- Practice Questions



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Lagrange's Method

First order Quasi-Linear PDE in standard form is called Lagrange's Equation

$$Pp + Qq = R$$

Where P, Q, R are function of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Step to Solve questions

Step-1 *Write down the PDE in standard form*

Step-2 *Write down the Auxiliary Equations*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{---------- (1)}$$

(I) (II) (III)

Step-3 Solve above auxiliary by

(i) Method of Grouping (I, II) , (II, III) , (I, III)

(ii) Method of Multipliers

Let two independent solutions are

$u(x, y, z) = c_1$ — (2)

$v(x, y, z) = c_2$ — (3)

Step-4 General Solution

$\phi(u, v) = 0$ **OR** $u = \phi(v)$ **OR** $v = \phi(u)$

Method of Multiplier

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Let Multipliers (P_1, Q_1, R_1)

Each fraction = $\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1}$

Type - 1

Select Multipliers in such a way that

$$PP_1 + QQ_1 + RR_1 = 0$$

Hence each fraction = $\frac{P_1dx + Q_1dy + R_1dz}{0}$

$$P_1dx + Q_1dy + R_1dz = 0$$

Integrate above equation to get solution

Type - 2

Select Multiplier in such a way that

Each fraction = $\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1}$

Numerator = Derivative of Denominator

Type - 1

Q.1 Solve the partial differential equation $(y^2+z^2)p - xyq = -zx$

$(y^2+z^2)p - xyq = -zx$

Here

$P = y^2+z^2$

$Q = -xy$

$R = -zx$

Aux. Eqn

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-zx} \quad \text{--- (1)}$$

I II III

(II, III)

$$\frac{dy}{-xy} = \frac{dz}{-zx}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log \frac{y}{z} = \log c_1$$

$$\boxed{\frac{y}{z} = c_1} \quad \text{--- (2)}$$

Taking x, y, z as Multiplier

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dy}{xy^2 + xz^2 - xy^2 - z^2x}$$

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dy}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$x^2 + y^2 + z^2 = 2c$$

$$x^2 + y^2 + z^2 = c_2 \quad \text{—— (3)}$$

General solution

$$\boxed{\phi\left(\frac{y}{z},\ x^2 + y^2 + z^2\right) = 0}$$

Q.2 Solve : $(y+zx)p-(x+yz)q=x^2-y^2$

(Sem. 2022)

$$(y+zx)p-(x+yz)q=x^2-y^2$$

Aux. Eqn

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2} \quad \text{—①}$$

Taking $(x, y, -z)$ as a Multipliers

$$\text{each fraction}=\frac{x\,dx+y\,dy-z\,dz}{xy+x^2z-xy-y^2z-zx^2+zy^2}$$

$$\text{each fraction}=\frac{x\,dx+y\,dy-z\,dz}{0}$$

$$x\,dx+y\,dy-z\,dz$$

$$\int x\,dx+\int y\,dy-\int z\,dz=0$$

$$\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=c$$

$$\boxed{x^2+y^2-z^2=c_1} \quad \text{—②}$$

Taking $(y, x, 1)$ as a Multiplier

$$\text{each fraction} = \frac{y\,dx + x\,dy + dz}{y^2 + xyz - x^2 - xyz + x^2 - y^2}$$

$$\text{each fraction} = \frac{y\,dx + x\,dy + dz}{0}$$

$$y\,dx + x\,dy + dz = 0$$

$$d(xy) + dz = 0$$

$$\int d(xy) + \int dz = 0$$

$$xy + z = c_2$$

General solution

$$\phi(x^2 + y^2 - z^2,\ xy + z) = 0$$

Q.3 Solve the following differential equations $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$. **(Sem. 2013)**

$x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Aux. Eq^n

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad \text{—①}$$

Taking $\left(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}\right)$ as a multiplier

$$\text{each fraction} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{y-z+z-x+x-y}$$

$$\text{each fraction} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

$$\int \frac{dx}{x^2} + \int \frac{dy}{y^2} + \int \frac{dz}{z^2} = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c$$

$$\boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1} \quad \text{—②}$$

Taking $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ as a Multiplier

$$\text{each fraction} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{xy - yz + yz - yx + zx - zy}$$

$$\text{each fraction} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \frac{dz}{z}$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2 \quad \text{—(2)}$$

General solution

$$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$$

Type - 2

Q.4 Solve the following partial differential equations : $x^2p + y^2q = (x+y)z$ (Sem. 2015)

$x^2p + y^2q = (x+y)z$

Aux. Eq^n

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

I II III

(I, II)

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} + c$$

$$\frac{1}{y} - \frac{1}{x} = c$$

$$\boxed{\frac{1}{x} - \frac{1}{y} = c_1} \quad -(2)$$

(III) — (1)

Taking $(1, -1, 0)$

each fraction $= \frac{dx - dy}{x^2 - y^2}$

each fraction $= \frac{dx - dy}{(x-y)(x+y)}$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{dx-dy}{(x-y)(x+y)}$$

I II III IV

(III, IV)

$$\frac{dz}{(x+y)z} = \frac{dx-dy}{(x-y)(x+y)}$$

$$\frac{dz}{z} = \frac{dx-dy}{x-y}$$

$$\int \frac{dz}{z} = \int \frac{dx-dy}{x-y}$$

$$\log z = \log(x-y) + \log c_2$$

$$\log z = \log(x-y) \times c_2$$

$$z = (x-y)\, c_2$$

$$\boxed{\frac{z}{x-y} = c_2} \quad -(3)$$

General solution

$$\boxed{\phi\left(\frac{1}{x}-\frac{1}{y}, \frac{z}{x-y}\right) = 0}$$

Q.5 Solve the following differential equations : $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$

$$(y^2+z^2-x^2)p - 2xyq + 2xz = 0$$

$$(y^2+z^2-x^2)p - 2xyq = -2xz$$

Aux. Eqn

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \quad \text{— (1)}$$

I II III

(II, III)

$$\frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\log y = \log z c_1$$

$$y = z c_1$$

$$\boxed{\frac{y}{z} = c_1} \quad \text{— (2)}$$

Taking (x, y, z) as a multiplier

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2}$$

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{-xy^2 - xz^2 - x^3}$$

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} = \frac{x\,dx + y\,dy + z\,dz}{-x^3 - xy^2 - xz^2}$$

I II III IV

III, IV

$$\frac{dz}{-2xz} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\int \frac{dz}{z} = \int \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\log z = \log(x^2+y^2+z^2) + \log c$$

$$\log z = \log(x^2+y^2+z^2) \times c$$

$$z = (x^2+y^2+z^2) \times c$$

$$c = \frac{z}{x^2+y^2+z^2}$$

$$\frac{1}{c} = \frac{x^2+y^2+z^2}{z}$$

$$c_2 = \frac{x^2+y^2+z^2}{z}$$

General solution

$$\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$

Q. 6 Solve : $(y+z)p+(z+x)q=x+y$

$$(y+z)p+(z+x)q=x+y$$

Aux. Eqⁿ

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y} \quad -(1)$$

Taking $(1,-1,0)$ as a multiplier

$$\text{each fraction}=\frac{dx-dy}{y+z-z-x}$$

$$=\frac{dx-dy}{-(x-y)}$$

Taking $(0,1,-1)$

$$\text{each fraction}=\frac{dy-dz}{z+x-x-y}$$

$$=\frac{dy-dz}{-(y-z)}$$

Taking $(1,1,1)$ as a multiplier

$$\text{each fraction}=\frac{dx+dy+dz}{y+z+z+x+x+y}$$

$$=\frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)} = \frac{dx+dx+dz}{2(x+y+z)}$$

I II III IV V VI

(IV, V)

$$\frac{dx-dy}{+(x-y)} = \frac{dy-dz}{+(y-z)}$$

$$\int \frac{dx-dy}{x-y} = \int \frac{dy-dz}{y-z}$$

$$\log(x-y) = \log(y-z) + \log c$$

$$\log(x-y) = \log(y-z) \times c$$

$$x-y = (y-z)\,c$$

$$\boxed{\frac{x-y}{y-z} = c} \quad -(2)$$

(IV, VI)

$$\frac{dx-dy}{-(x-y)} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{dx-dy}{(x-y)} + \frac{dx+dy+dz}{2(x+y+z)} = 0$$

$$\int \frac{dx-dy}{x-y} + \frac{1}{2}\int \frac{dx+dy+dz}{x+y+z} = 0$$

$$\log(x-y) + \frac{1}{2}\log(x+y+z) = \log c$$

$$2\log(x-y) + \log(x+y+z) = 2\log c$$

$$\log(x-y)^2 + \log(x+y+z) = \log c_2$$

$$\log(x-y)^2 \times (x+y+z) = \log c_2$$

$$\boxed{(x-y)^2(x+y+z) = c_2} \quad - (3)$$

General solution

$$\boxed{\phi\left[\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right] = 0}$$

Practice Q.1 Solve the following differential equation : $9mz - ny)p + (n\,x - lz)q = ly - m\,x$

Practice Q.2 Solve the following differential equations : $(x^2 - yz)p + (y^2 - z\,x)q = z^2 - xy$

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Thank You