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DPP-5 Maths-IV Unit-2

Topic- Solution of one dimmensional Wave Equation

Q-1: Show that the wave equation $c^2 \frac{\delta^2 y}{\delta x^2} = \frac{\delta^2 y}{\delta t^2}$ can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched $x=0$ and $x=l$ are given by $y=f(x)$ and $\frac{\delta y}{\delta t} = g(x)$, determine constants in series solution?

The equation of string is

$$\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2}$$

Length of string $(l) = l$

Boundry conditions, At $x=0$, $y=0$ BC-1
At $x=l$, $y=0$ BC-2

Initial conditions, At $t=0$, $y=f(x)$ IC-1
At $t=0$, $\frac{\delta y}{\delta t} = g(x)$ IC-2

Most suitable solution,

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

pply Boundry condition-1,

$$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$$

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$C_1 = 0$

Updated solution :

$$y = C_2 \sin px \, (C_3 \cos cpt + C_4 \sin cpt)$$

Apply BC-2;

$$0 = C_2 \sin pl \, (C_3 \cos cpt + C_4 \sin cpt)$$

$$\sin pl = 0$$
$$pl = n\pi$$
$$p = \frac{n\pi}{l}$$

Updated solution :

$$y = C_2 \sin \frac{n\pi}{l} x \left(C_3 \cos c \frac{n\pi}{l} t + C_4 \sin \frac{n\pi c}{l} t \right)$$

Apply IC-1;

$$f(x) = C_2 \sin \frac{n\pi}{l} x \cdot C_3$$

$$f(x) = C_2 C_3 \sin \frac{n\pi x}{l}$$

$$f(x) = \sum_{1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

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Apply IC-2;

$$\therefore a_n = C_2 C_3$$
$$\therefore b_n = C_2 C_4$$

$$\frac{\partial y}{\partial t} = \frac{\pi c}{l} \sum_{n=1}^{\infty} \left(-n a_n \sin \frac{n\pi ct}{l} + n b_n \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

$$g(x) = n b_n \sin \frac{n\pi x}{l} \times \frac{\pi c}{l}$$

$$\frac{n\pi c}{l} b_n = \frac{2}{l} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

Hence the required solution,

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}}$$

where, $$a_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

Q-2: A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $F(x) = \mu x(l-x)$, μ is a constant and then released. Find the displacement $y(x,t)$ of any point x of the string at any time $t>0$.

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The equation of string is

$$\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2}$$

Boundry conditions;

At $x = 0$, $y = 0$ BC-1

At $x = l$, $y = 0$ BC-2

Initial conditions;

At $t = 0$, $\frac{\delta y}{\delta t} = 0$ IC-1

At $t = 0$, $y = \mu x (l - x)$ IC-2

Most suitable solution,

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

Apply BC-1;

$$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$$

$$\boxed{C_1 = 0}$$

Updated solution,

$$y = C_2 \sin px \ (C_3 \cos cpt + C_4 \sin cpt)$$

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Apply BC-2;

$$0 = C_2 \sin pl \left(C_3 \cos cpt + C_4 \sin cpt\right)$$

$$\sin pl = 0$$
$$pl = n\pi$$
$$p = \frac{n\pi}{l}$$

Updated solution:-

$$y = C_2 \sin \frac{n\pi}{l} x \left(C_3 \cos c\frac{n\pi}{l} t + C_4 \sin \frac{n\pi c t}{l}\right)$$

Apply IC-1;

$$\frac{\delta y}{\delta t} = \frac{n\pi c}{l}\left[-C_3 \sin \frac{n\pi c t}{l} + C_4 \cos \frac{n\pi c t}{l}\right] \cdot C_2 \sin \frac{n\pi x}{l}$$

$$t = 0, \quad \frac{\delta y}{\delta t} = 0$$

$$0 = \frac{n\pi c}{l} C_2 C_4 \sin \frac{n\pi x}{l}$$

$$\boxed{C_4 = 0}$$

Updated solution,

$$y(x,t) = C_3 C_2 \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

Apply IC-2

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$$u(lx-x^2) = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Using half range sine series

$$b_n = \frac{2}{l}\int_0^l u(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2u}{l}\left[\left\{(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)\right\}_0^l - \int_0^l (l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)dx\right]$$

$$= \frac{2u}{l}\left[\frac{l}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2u}{n\pi}\left[\left\{(l-2x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l - \int_0^l (-2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\,dx\right]$$

$$= \frac{2u}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4ul}{n^2\pi^2}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l$$

$$= \frac{4ul^2}{n^3\pi^3}\left(-\cos n\pi + 1\right)$$

$$= \frac{4ul^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

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$$b_n = \frac{4ul^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$y(x,t) = \frac{4ul^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

For even $\Rightarrow$ $y = 0$

For odd $\Rightarrow$ $y = \frac{8ul^2}{\pi^3}\sum_{n=1}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}$

Ans

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