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DPP-2 Maths-IV Unit-2

Topic:- Method of separation of Variables

Q-1:- Solve by the method of separation of variables.
$3\frac{\delta u}{\delta x} + 2\frac{\delta u}{\delta y} = 0$; $u(x,0) = 4e^{-x}$.

Sol:- $3\frac{\delta u}{\delta x} + 2\frac{\delta u}{\delta y} = 0$ — (1)

Let $u = X(x)Y(y)$
$u = XY$ — (2)

$\frac{\delta u}{\delta x} = X'Y$

$\frac{\delta u}{\delta y} = XY'$

Put in eq (1)

$3X'Y + 2XY' = 0$

Now divide by XY

$3\frac{X'}{X} + 2\frac{Y'}{Y} = 0$

$3\frac{X'}{X} = -2\frac{Y'}{Y} = K$ (say) — (3)

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$$\frac{3X'}{X} = K$$

$$3X' = XK$$

$$(3D - K)X = 0$$

Auxiliary equation $\Rightarrow$ $3D - K = 0$

$$D = \frac{K}{3}$$

$$\boxed{C.F. = C_1 e^{Kx/3}}$$

$$PI = 0$$

$$X = C.F + PI$$

$$\boxed{X = C_1 e^{Kx/3}}$$

From eq 3.

$$\frac{-2Y'}{Y} = K$$

$$-2Y' = YK$$

$$YK + 2Y' = 0$$

$$(2D + K)Y = 0$$

Auxiliary eq $\Rightarrow$ $2D + K = 0$

$$D = -K/2$$

$$\boxed{C.F. = C_2 e^{-Ky/2}}$$

$$PI = 0$$

$$\boxed{Y = C_2 e^{-Ky/2}}$$

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Put value of X and Y in eq 2:

$u = XY$

$u = C_1 e^{Kx/3} C_2 e^{-Ky/2}$

$$u = C_1 C_2 e^{K\left(\frac{x}{3} - \frac{y}{2}\right)} \quad \text{— (4)}$$

$u(x,y)$

Apply Boundry condition $y = 0$

$u = 4e^{-x}$

$4e^{-x} = C_1 C_2 e^{Kx/3}$

where

$C_1 C_2 = 4$

$$-1 = \frac{K}{3}$$

$K = -3$

Put $C_1 C_2$ and K in eq — (4)

$$u(x,y) = 4e^{-3\left(\frac{x}{3} - \frac{y}{2}\right)}$$

$$u(x,y) = 4e^{-x + 3y/2}$$

Ans

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Q-2 : Solve by the method of separation of variables.
$4\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = 3u$, $u(0,y) = 4e^{-y} - e^{-5y}$

Sol : $4\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = 3u$ — (1)

Let $u = X(x)Y(y)$

$u = XY$ — (2)

$\frac{\delta u}{\delta x} = X'Y$

$\frac{\delta u}{\delta y} = XY'$

Put value in eq (1)

$4X'Y + XY' = 3XY$

$4\frac{X'}{X} = 3 - \frac{Y'}{Y} = K$ (say)

$4\frac{X'}{X} = K$

$4X' = KX$

$(4D - K)X = 0$

$D = K/4$

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$$C.F. = C_1 e^{Kx/4}$$

$$PI = 0$$

$$X = C_1 e^{Kx/4}$$

$$3 - \frac{y'}{y} = K$$

$$3y - y' = Ky$$

$$y' + Ky - 3y = 0$$

$$(D + K - 3)y = 0$$

$$D = 3 - K$$

$$C.F. = C_2 e^{(3-K)y}$$

$$PI = 0$$

$$y = C_2 e^{(3-K)y}$$

From equation (2)

$$u = C_1 C_2 e^{\frac{Kx}{4} + (3-K)y}$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n e^{Kx/4 + (3-K)y} \longrightarrow (3)$$

Apply Boundry conditions: Put $x = 0$

$$4e^{-y} - e^{-5y} = \sum_{n=1}^{\infty} b_n e^{(3-K)y}$$

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$4e^{-y} - e^{-5y} = b_1 e^{(3-K)y} + b_2 e^{(3-K)y} + \ldots\ldots\ldots$

By comparison

$b_1 = 4$ $\quad$ $b_2 = -1$

$-1 = 3-K$ $\quad$ $-5 = 3-K$

$-4 = -K$ $\quad$ $K = 8$

$K = 4$

From ③

$u(x,y) = b_1 e^{\frac{Kx}{4}+(3-K)y} + b_2 e^{\frac{Kx}{4}+(3-K)y}$

$\boxed{u(x,y) = 4e^{x-y} - e^{2x-5y}}$

Ans

Q-3- Use the method of separation of variables to solve the equation $\frac{\delta^2 v}{\delta x^2} = \frac{\delta v}{\delta t}$ given that $v = 0$ when $t \to \infty$ as well as $v = 0$ at $x = 0$ and $x = l$.

Sol- $\frac{\delta^2 v}{\delta x^2} = \frac{\delta v}{\delta t}$ — ①

Let $v = X(x)T(t)$ —

$v = XT$ — ②

$\frac{\delta v}{\delta t} = XT'$, $\frac{\delta v}{\delta x} = X'T$, $\frac{\delta^2 v}{\delta x^2} = X''T$

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Put in eq (1)

$$X''T = XT'$$

Divide by XT

$$\frac{X''}{X} = \frac{T'}{T} = -p^2 \text{ (say)}$$

$$\frac{X''}{X} = -p^2$$

$$X'' = -Xp^2$$

$$X'' + Xp^2 = 0$$

$$(D^2 + p^2)X = 0$$

$$D^2 = -p^2$$
$$D = \pm ip$$

$$C.F. = e^{0x}(C_1 \cos px + C_2 \sin px)$$

$$\boxed{C.F. = C_1 \cos px + C_2 \sin px}$$

$$PI = 0$$

$$\boxed{X = C_1 \cos px + C_2 \sin px}$$

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$$\frac{T'}{T} = -p^2$$

$$T' = -Tp^2$$

$$T' + Tp^2 = 0$$

$$(D + p^2)T = 0$$

$$D = -p^2$$

$$\boxed{C.F. = C_3 e^{-p^2 t}}$$

$$PI = 0$$

$$\boxed{T = C_3 e^{-p^2 t}}$$

$$v = (C_1 \cos px + C_2 \sin px) C_3 e^{-p^2 t} \quad \text{—③}$$

Apply Boundry conditions,

$$v = 0, \quad x = 0$$

$$0 = C_1 C_3 e^{-p^2 t}$$

$$C_3 \neq 0$$

$$\boxed{C_1 = 0}$$

Now put in eq 3

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Updated solution :

$$v = C_2 C_3 \sin px \, e^{-p^2 t}$$

Now $v = 0$, $x = l$

$$0 = C_2 C_3 \sin pl \, e^{-p^2 t}$$

$$\sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Updated solution :

$$v = C_2 C_3 \sin \frac{n\pi}{l} x \, e^{-\frac{n^2\pi^2}{l^2} t}$$

General solution will be :

$$\boxed{v = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{l^2} t} \sin \frac{n\pi x}{l}}$$

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Q-4- Solve the P.D.E by separation of variables method,

$$u_{xx} = u_y + 2u \;,\; u(0,y) = 0,\; \frac{\delta}{\delta x} u(0,y) = 1 + e^{-3y}$$

Sol:- Let $u = XY$

$$\frac{\delta u}{\delta y} = XY' \;,\; \frac{\delta u}{\delta x} = X'Y \;,\; \frac{\delta^2 u}{\delta x^2} = X''Y$$

$$u_{xx} = u_y + 2u$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta u}{\delta y} + 2u \qquad \text{— (1)}$$

$$X''Y = XY' + 2XY$$

Divide by XY

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = k \text{ (say)}$$

$$\frac{X''}{X} = k$$

$$X'' - kX = 0$$
$$(D^2 - k)X = 0$$

Auxiliary Equation $\Rightarrow$ $D^2 - k = 0$

$$D = \pm\sqrt{k}$$

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$$C.F. = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

$$PI = 0$$

$$\boxed{X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}}$$

$$\frac{y'}{y} + 2 = k$$

$$\frac{y'}{y} = k-2$$

$$y' - y(k-2) = 0$$

$$[D-(k-2)]y = 0$$

Auxiliary Equation $\Rightarrow$ $D-(k-2) = 0$

$$D = k-2$$

$$C.F. = C_3 e^{(k-2)y}$$

$$PI = 0$$

$$\boxed{Y = C_3 e^{(k-2)y}}$$

$$u(x,y) = \left(C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}\right) C_3 e^{(k-2)y}$$

Apply Boundry condition, $x = 0$, $u = 0$.

$$0 = (C_1 + C_2) C_3 e^{(k-2)y}$$

$$C_1 + C_2 = 0$$

$$C_2 = -C_1$$

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Updated solution :

$$u(x,y) = C_1 C_3 \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right) e^{(k-2)y}$$

$$\frac{\delta u}{\delta x} = C_1 C_3 \sqrt{k} \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right) e^{(k-2)y}$$

Put Boundary Condition, $x = 0$

$$\left(\frac{\delta u}{\delta x}\right)_{x=0} = C_1 C_3 \sqrt{k} (2) e^{(k-2)y}$$

$$1 + e^{-3y} = \Sigma b_n e^{(k-2)y} \qquad \{ b_n = 2 C_1 C_3 \sqrt{k} \{$$

Comparing the coefficients,

i) $b_1 = 1$, $k - 2 = 0$

$k = 2$

$$2 C_1 C_3 \sqrt{k} = 1$$

$$C_1 C_3 = \frac{1}{2\sqrt{2}}$$

ii) $b_3 = -1$, $k - 2 = -3$

$k = -1$

$$2 C_1 C_3 \sqrt{k} = 1$$

$$C_1 C_3 = \frac{1}{2i}$$

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$$u(x,y) = \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}x} - e^{-\sqrt{2}x}\right) + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)e^{-3y}$$

$$\boxed{u(x,y) = \frac{1}{\sqrt{2}} \sinh\sqrt{2}x + e^{-3y}\sin x}$$

Ans

$\because \frac{e^{ax} - e^{-ax}}{2} = \sinh ax$

$\because \frac{e^{ix} - e^{-ix}}{2i} = \sin x$

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