ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- One Dimensional Heat Equation (Part – 3)
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

One Dimensional Heat Equation : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

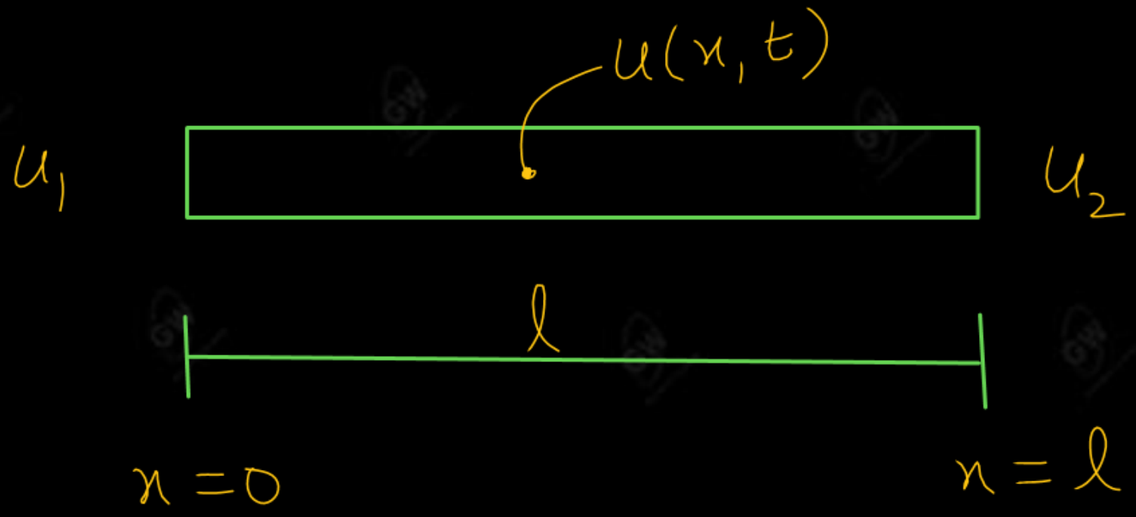
On the Basis of Boundary conditions, there are three types of question

Type -1: When both ends of rod are at $0^o C$

Type -2: When both ends of rod are perfectly insulated

Type -3: When both ends of rod are kept at different temperature.

Type -3 : When both ends of rod are kept at different temperature.



$$u_s = a + bx$$

$$u_t = (c_1 \cos px + c_2 \sin px) c_3 e^{-cp^2 t}$$

complete solution

$$u(x,t) = u_s(x) + u_t(x,t)$$

$u_s(x)$: *Steady state*

$u_t(x,t)$: *Transient state*

Q. 1: A bar of 10 cm length with insulated sides A and B are kept and 20°C and 40°C respectively until steady state conditions prevail. The temperature at A is then suddenly varied to 50°C and the same instant at B, lowered at 10°C. Find the subsequent temperature at any point of the bar at any time. *(A.K.T.U. 2017)*

One-dimensional heat equation

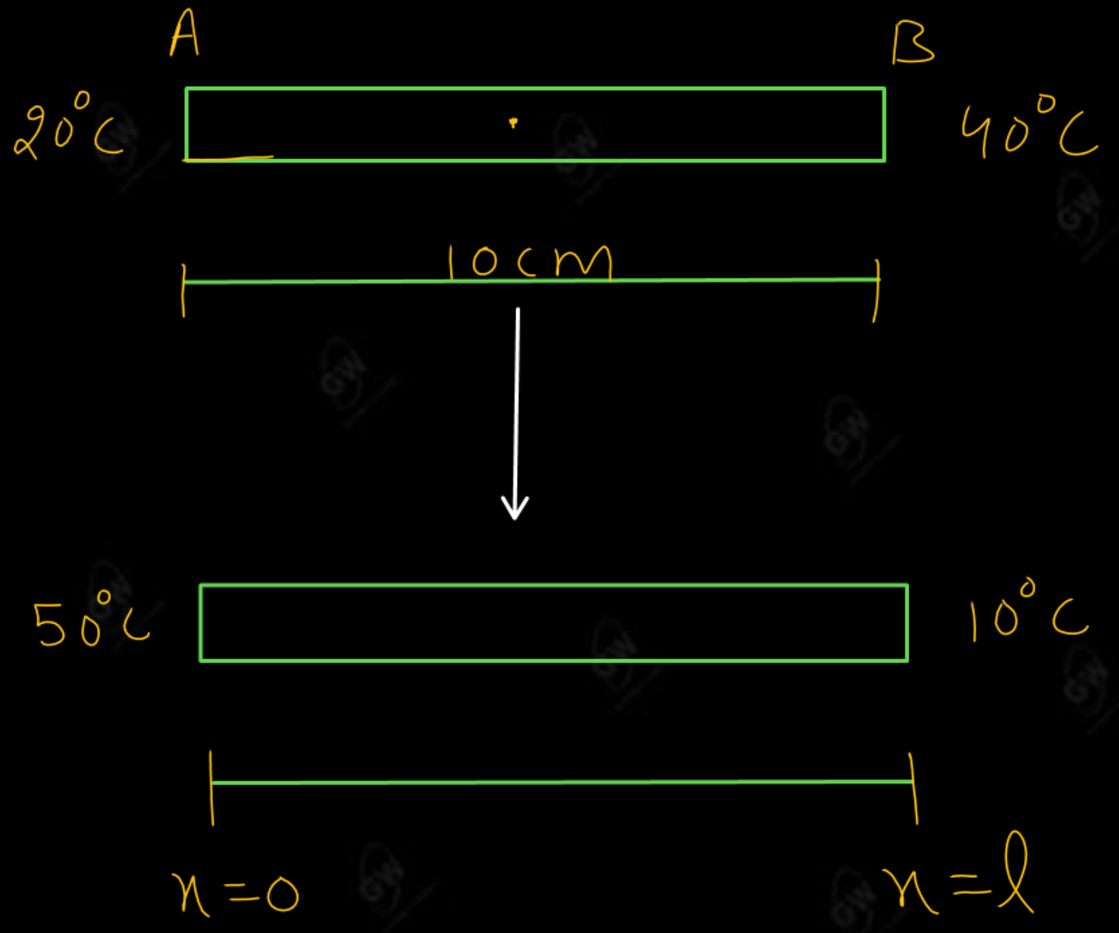
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— ①}$$

length of bar $(l) = 10\,cm$

Boundary conditions

At $x = 0$, $u = 50^\circ$ B(-1

At $x = 10\,cm$, $u = 10^\circ C$ B(-2



Initial condition

$$u = T_A + \left(\frac{T_B - T_A}{10}\right)x$$

$$u = 20 + \left(\frac{40-20}{10}\right)x$$

$$u = 20 + 2x$$

At $t = 0$, $u = 20 + 2x$

Comple solution of (1)

$$u(x,t) = u_s(x) + u_s(x,t)$$

$$u(x,t) = (a + bx) + (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2 p^2 t}$$ — (8)

Steady state solution

$$u_s(x) = T_A + \left(\frac{T_B - T_A}{10}\right)x$$

$$= 50 + \frac{(10-50)}{10}x$$

$$\boxed{u_s(x) = 50 - 4x}$$

Put in (8)

$$u(x,t) = (50-4x) + (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2p^2t} \quad \text{— (9)}$$

Apply BC-1

$$\cancel{50} = \cancel{50} + C_1 \times C_3 e^{-c^2p^2t}$$

$$0 = C_1 C_3 e^{-c^2p^2t}$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$u(x,t) = (50-4x) + C_2 \sin px \times C_3 e^{-c^2p^2t} \quad \text{— (10)}$$

Apply BC-2

$$10 = 50 - 40 + C_2 \sin 10p \times C_3 e^{-c^2p^2t}$$

$$\cancel{10} = \cancel{10} + C_2 \sin 10p \times C_3 e^{-c^2p^2t}$$

$$0 = C_2 \sin 10p \times C_3 e^{-c^2p^2t}$$

$$\Rightarrow \sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

Updated solution

$$u(x,t) = (50-4x) + C_2 \sin\frac{n\pi}{10}x \times C_3 e^{-\frac{c^2 n^2 \pi^2}{100}t} \quad \text{—— (11)}$$

Most general solution

$$u(x,t) = (50-4x) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x \times e^{-\frac{c^2 n^2 \pi^2}{100}t} \quad \text{—— (12)}$$

Apply initial condition

$$20+2x = 50-4x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x \times 1$$

$$-30+6x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Apply Half range sine series

Formula

$$f(x) = \sum b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{\cancel{2}}{\cancel{10}_5}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

I II

$$b_n = \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$b_n = \frac{1}{5}\left[\frac{-30\cos n\pi}{\frac{n\pi}{10}} - 0 - \left(\frac{-30(-1)}{\frac{n\pi}{10}} - 0\right)\right]$$

$$b_n = \frac{1}{5}\left[\frac{-300\cos n\pi}{n\pi} - \frac{300}{n\pi}\right]$$

$$b_n = \frac{-\overset{60}{\cancel{300}}}{\cancel{5}\, n\pi}\left(\cos n\pi + 1\right)$$

$$b_n = -\frac{60}{n\pi}\left(1+\cos n\pi\right)$$

Put b_n in (12)

$$u(x,t) = (50-4x) + \sum_{n=1}^{\infty} \frac{-60}{n\pi}\left(1+\cos n\pi\right) \times \sin\frac{n\pi x}{10} \times e^{-\frac{c^2 n^2 \pi^2}{100} t}$$

$$u(x,t) = (50-4x) - \frac{60}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(1+\cos n\pi)\sin\frac{n\pi x}{10}\, e^{-\frac{c^2 n^2 \pi^2}{100}t}$$

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Topic : One Dimensional Heat Equation

Q.1. (i) An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t. [G.B.T.U. (A.G.) 2011, U.K.T.U. 2011]

(ii) Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

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