ENGINEERING MATHEMATICS

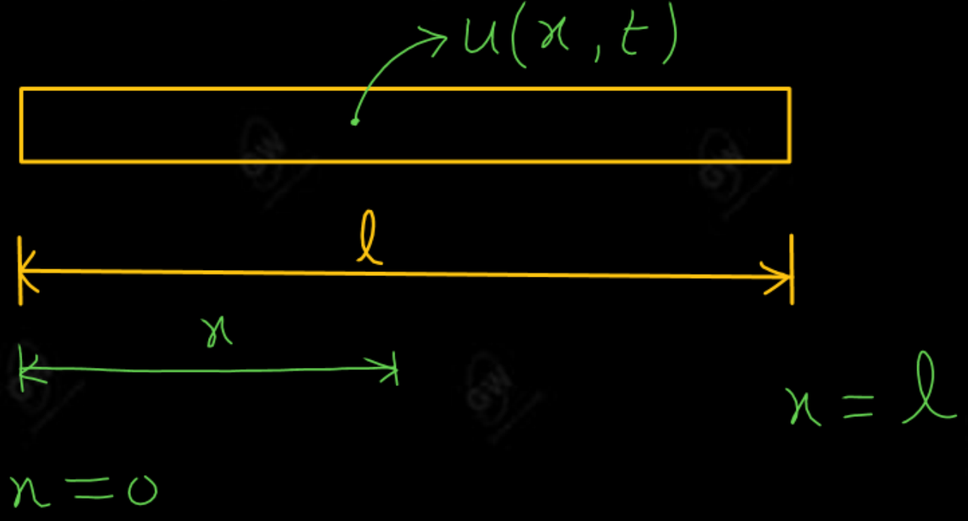
UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- One Dimensional Heat Equation (Part – 1)
- PYQ
- DPP

One Dimensional Heat Equation

Consider the flow of heat in a homogeneous rod of length l and variation of temperature with position and time.



Assumptions

(i) Rod has uniform cross-section

(ii) Rod has constant density

(iii) Lateral surface of rod is insulated

(iv) Heat flow only in one direction

(v) The rod is sufficiently thin so that the temperature is same at all points of any cross-section area of rod

Let $u(x, t)$ be the temperature distribution of the cross-section at the point x and at any time t

Mathematically

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This partial differential equation is known as one-dimensional Heat equation

Q. 1:- *Find the solution of One – dimensional heat equation by the method of separation of variables*

One - Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

Let $u = XT$ —— (2)

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = TX''$$

Put these values in (1)

$$XT' = c^2 T X''$$

Divide by XT

$$\frac{XT'}{XT} = \frac{c^2 T X''}{XT}$$

$$\frac{T'}{T} = c^2 \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2} \times \frac{T'}{T} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{—(3)}$$

$$\frac{1}{c^2} \times \frac{T'}{T} = K$$

$$T' = c^2 KT$$

$$DT - c^2 KT = 0$$

$$(D - c^2 K)T = 0 \quad \text{—(4)}$$

CASE-1: When $K = 0$

$$D^2X = 0$$

$$D^2 = 0$$

$$D = 0, 0$$

$$X = c_1 + c_2 x$$

$$DT = 0$$

$$D = 0$$

$$T = c_3 e^{0t}$$

$$T = c_3$$

Put x and T in (2)

$$u(x,t) = (C_1 + C_2 x) C_3$$

$$u(x,t) = C_1 C_3 + C_2 C_3 x$$

$$\boxed{u(x,t) = a + bx} \quad \text{—(5)}$$

Time-independent solution

$\Rightarrow$ steady-state solution

CASE-2: When $K = p^2$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D^2 = p^2$$

$$D = \pm p$$

$$X = C_1 e^{px} + C_2 e^{-px}$$

$$(D - p^2 c^2)T = 0$$

$$D - p^2 c^2 = 0$$

$$D = p^2 c^2$$

$$T = C_3 e^{p^2 c^2 t}$$

Put these values in (2)

$$\boxed{u(x,t) = (C_1 e^{px} + C_2 e^{-px}) C_3 e^{p^2 c^2 t}}$$

— (6)

CASE-3: When $K = -p^2$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D = \sqrt{-p^2}$$

$$D = \pm ip$$

$D = 0 + iP$

$X = C_1 \cos Px + C_2 \sin Px$

$(D + c^2p^2)T = 0$

$D + c^2p^2 = 0$

$D = -c^2p^2$

$T = C_3 e^{-c^2p^2t}$

Put these values in (2)

$$u(x,t) = (C_1 \cos Px + C_2 \sin px)\, C_3 e^{-c^2p^2t} \quad \text{— (7)}$$

Most suitable solution

$$u(x,t) = (C_1 \cos Px + C_2 \sin px)\, C_3 e^{-c^2p^2t} \quad \text{— (8)}$$

steady state solution

$u(x,t) = a + bx$

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comple solution

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t} + a + bx$$

$$u(x,t) = u_t(x,t) + u_s(x)$$

Transient solution

steady state solution

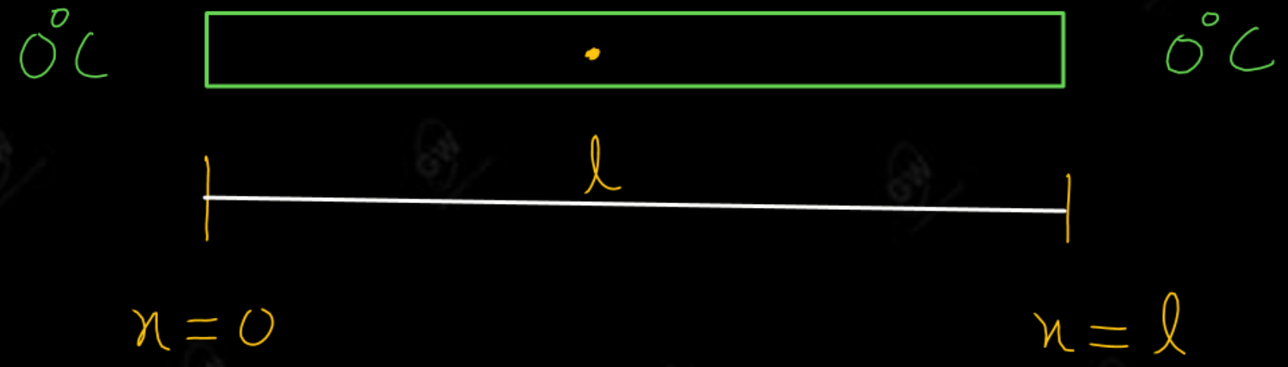
On the Basis of Boundary conditions

Type -1: When both ends of rod are at 0^oC

Type -2: When both ends of rod are perfectly insulated

Type -3: When both ends of rod are kept at different temperature.

Type -1 : When both ends of rod are at 0^oC



$$u(x,t) = u_s(x) + u_t(x,t)$$

Steady state | Transient state

Boundary Conditions

$$At\ x = 0,\ u = 0 \Rightarrow u(0,t) = 0^oC \quad BC-1$$

$$At\ x = l,\ u = 0 \Rightarrow u(l,t) = 0^oC \quad BC-2$$

$u_s(x) = a + bx$

Apply BC-1

$0 = a + 0$

$a = 0$

$u_s(x) = bx$

$0 = bl$

$b = 0$

$\Rightarrow \boxed{u_s(x) = 0}$

$u(x,t) = u_t(x,t)$

$$\boxed{u(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2p^2t}}$$

Q. 1:- *Find the temperature in a bar of length 2 m whose ends are kept at zero and lateral surface insulated if the initial temperature is* $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.

(U.P.T.U. 2015)

One-Dimensional Heat equation

$$\frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——— (1)}$$

Length of rod $(l) = 2m$

Boundary conditions

At $x = 0$, $u = 0$ B(-1

At $x = 2$, $u = 0$ B(-2

Initial condition

At $t = 0$, $u = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Most suitable solution

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}$$

└ (8)

Apply B(-1

$$0 = c_1 c_3 e^{-c^2 p^2 t}$$

$\Rightarrow C_1 = 0$

Updated solution

$$u(x,t) = C_2 \sin px \times C_3 e^{-c^2 p^2 t}$$

— (9)

Apply BC-2

$$0 = C_2 \sin 2p \times C_3 e^{-c^2 p^2 t}$$

$\Rightarrow \sin 2p = 0$

$\sin 2p = \sin n\pi$

$2p = n\pi$

$p = \frac{n\pi}{2}$

Updated solution

$$u(x,t) = C_2 \sin \frac{n\pi}{2} x \times C_3 e^{-\frac{c^2 n^2 \pi^2 t}{4}}$$

$$u(x,t) = C_2 C_3 \sin \frac{n\pi}{2} x \times e^{-\frac{c^2 n^2 \pi^2}{4} t} \quad — (10)$$

Most general solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x \times e^{-\frac{c^2 n^2 \pi^2 t}{4}} \quad — (11)$$

Apply IC

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = b_1 \sin\frac{\pi x}{2} + b_2 \sin\pi x + b_3 \sin\frac{3\pi x}{2} + b_4 \sin 2\pi x + b_5 \sin\frac{5\pi x}{2} + -$$

on comparison

$b_1 = 1$ $\quad b_4 = 0$

$b_2 = 0$ $\quad b_5 = 3$

$b_3 = 0$

From (11)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}x \times e^{\frac{-c^2n^2\pi^2t}{4}}$$

$$u(x,t) = b_1 \sin\frac{\pi x}{2} \times e^{\frac{-c^2\pi^2t}{4}} + b_5 \sin\frac{5\pi x}{2}\, e^{\frac{-c^2 25\pi^2 t}{4}}$$

$$u(x,t) = e^{\frac{-c^2\pi^2t}{4}} \sin\frac{\pi x}{2} + 3e^{\frac{-25c^2\pi^2t}{4}} \sin\frac{5\pi x}{2}$$

Q. 2:- Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$. (G.B.T.U. 2012)

One-Dimensional Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \quad (1)$$

length of rod = 2m

Boundary conditions

At $x = 0$, $u = 0$ B.C-1

At $x = 2m$, $u = 0$ B.C-2

Initial condition

At $t = 0$, $u = f(x) = 100x$

Most suitable solution

$$u(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2 p^2 t}$$

—— (8)

Apply B.C-1

$$0 = C_1 C_3 e^{-c^2 p^2 t}$$

$\Rightarrow C_1 = 0$

Updated solution

$$u(x,t) = C_2 \sin px \times C_3 e^{-c^2 p^2 t}$$

(9)

Apply BC-2

$$0 = C_2 \sin 2p \times C_3 e^{-c^2 p^2 t}$$

$\Rightarrow \sin 2p = 0$

$\sin 2p = \sin n\pi$

$2p = n\pi$

$p = \frac{n\pi}{2}$

Updated solution

$$u(x,t) = C_2 \sin \frac{n\pi}{2} x \times C_3 e^{-c^2 p^2 t}$$

(10)

Most General solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x \times e^{-c^2 p^2 t}$$

(11)

Apply I C

$$100x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

Using Half-Range sine-series

$$b_n = \frac{2}{2}\int_0^2 100x \sin\frac{n\pi x}{2}\,dx$$

$$b_n = 100\int_0^2 x \sin\frac{n\pi x}{2}\,dx$$

I II

$$b_n = 100\left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^2$$

$$b_n = 100\left[\frac{-2\cos n\pi}{\frac{n\pi}{2}} - 0 - (0-0)\right]$$

$$b_n = 100\left(\frac{-4\cos n\pi}{n\pi}\right)$$

$$b_n = -\frac{400}{\pi}\left(\frac{\cos n\pi}{n}\right)$$

Put b_n in (11)

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{-400}{\pi}\right)\left(\frac{\cos n\pi}{n}\right) \sin\frac{n\pi x}{2} \times e^{-c^2p^2t}$$

$$u(x,t) = \frac{-400}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos n\pi}{n}\right) \sin\frac{n\pi x}{2} \times e^{-c^2p^2t}$$

Where $\cos n\pi = (-1)^n$

DPP- 6

Topic : One Dimensional Heat Equation

Q.1. A *rod of length l with insulated sides is initially at a uniform temperature* u_0. *Its ends are suddenly cooled to 0°C and are kept* at *that temperature. Find the temperature function* $u(x, t)$. *(U.P.T.U. 2015)*

Q.2. Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $u(0, t) = 0, u(l, t) = 0\ (t > 0)$ and the initial condition $u(x, 0) = x$: l being the length of the bar. *(U.P.T.U. 2015)*

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