ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Complex Fourier transform***
- PYQ
- DPP

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or $f(p)$ and defined as

$$F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x) e^{ipx} dx$$

F(s)

$i = \sqrt{-1}$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(p) e^{-ipx} dp$$

Some Important Points and Formulae

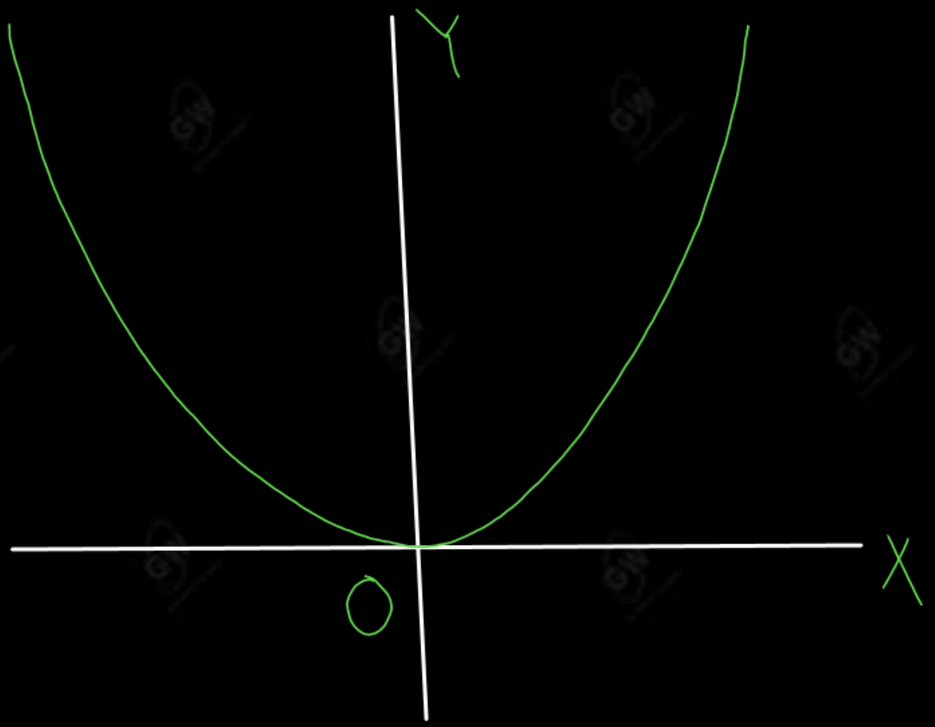
1. Even and Odd Function :

(i) Even Function : A function is said be even if

$$f(-x) = f(x)$$

Note: Graph of even function is symmetrical about Y-axis

$f(x) = x^2$

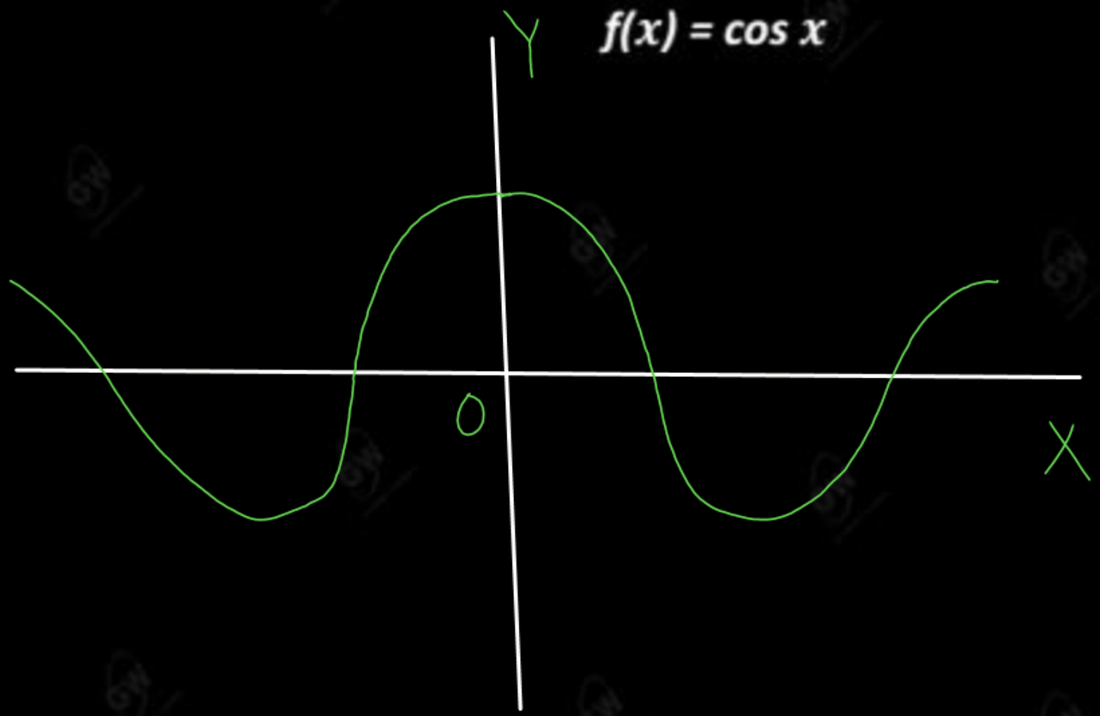


Example

(1) $f(x) = \cos x$

(2) $f(x) = x^2$

$f(x) = \cos x$



(ii) Odd Function : A function is said be Odd if

$$f(-x) = -f(x)$$

Example

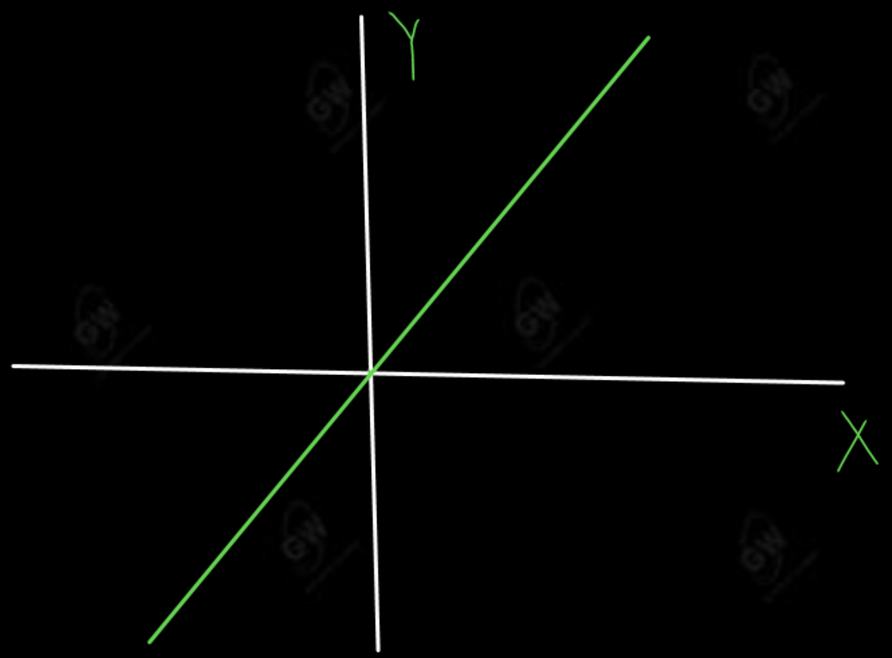
(a) $f(x) = \sin x$

(b) $f(x) = \sin^3 x$

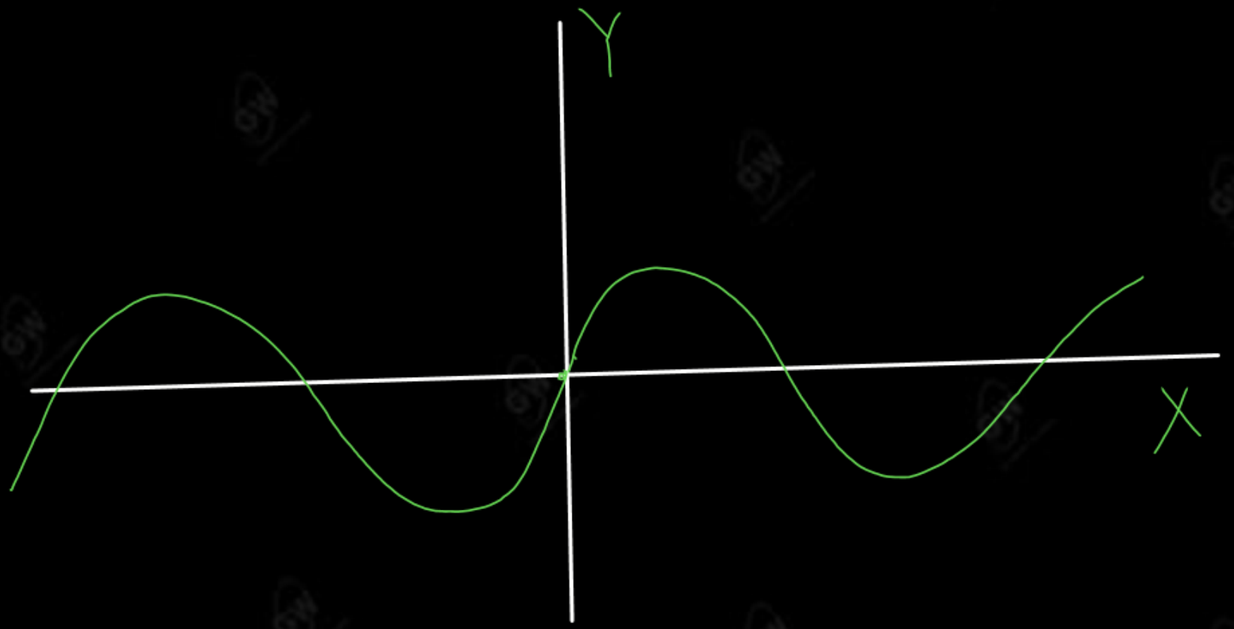
(c) $f(x) = x$

Note: The graph $f(x)$ of an odd function is symmetrical about origin and in opposite quadrant.

$f(x) = x$



$f(x) = \sin x$



Also Remember

(i) Even function × Even function = Even function

(ii) Odd function × Odd function = Even function

(iii) Even function × Odd function = Odd function

(iv) Odd function × Even function = Odd function

2 An important Property of Definite integral

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \qquad \text{If } f(x) \text{ is an even function}$$

$$= 0 \qquad \text{If } f(x) \text{ is an odd function}$$

③ Integration By-Parts

$$\int_{I\ II} uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

④ $$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

⑤ $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

⑥ $$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

GATEWAY CLASSES

NOTE:

① $|x| \leq a$

$-a \leq x \leq a$

② $|x| < a$

$-a < x < a$

③ $|x| \geq a$

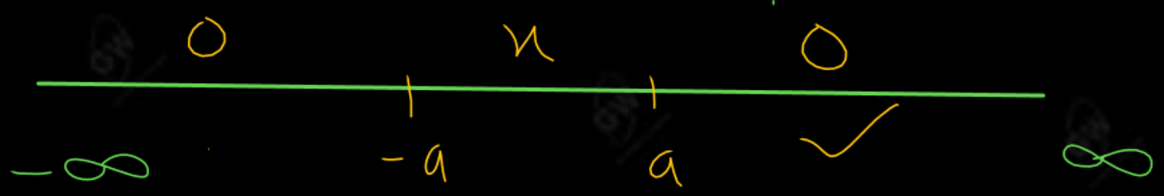
$x \geq a$

$x \leq -a$

④ $|x| > a$

$x > a$

$x < -a$



Q.1. *Find the Fourier Transform of* $F(x) = \begin{cases} x, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$

$$F(x) = \begin{cases} x & -a < x < a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{-a} F(x) e^{ipx} dx + \int_{-a}^{a} F(x) e^{ipx} dx + \int_{a}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = 0 + \int_{-a}^{a} x\, e^{ipx} dx + 0$$

$$f(p) = \int_{-a}^{a} x\, e^{ipx} dx$$

I II

$$f(p) = \left[x\left(\frac{e^{ipx}}{ip}\right) - 1\left(\frac{e^{ipx}}{i^2 p^2}\right) \right]_{-a}^{a}$$

$$f(p) = \frac{a e^{ipa}}{ip} - \frac{e^{ipa}}{i^2 p^2} - \left(\frac{-a e^{-ipa}}{ip} - \frac{e^{-ipa}}{i^2 p^2} \right)$$

$$f(p) = \frac{a e^{ipa}}{ip} + \frac{e^{ipa}}{p^2} + \frac{a e^{-ipa}}{ip} - \frac{e^{-ipa}}{p^2}$$

$$f(p) = \frac{a}{ip}\left(e^{ipa} + e^{-ipa} \right) + \frac{1}{p^2}\left(e^{ipa} - e^{-ipa} \right)$$

$$f(p) = \frac{2a}{ip}\left(\frac{e^{ipa} + e^{-ipa}}{2} \right) + \frac{2i}{p^2}\left(\frac{e^{ipa} - e^{-ipa}}{2i} \right)$$

$$f(p) = \frac{2a}{ip}\cos ap + \frac{2i}{p^2}\sin ap$$

$$f(p) = \frac{2i}{p^2}\left(\frac{ap\cos ap}{i^2} + \sin ap \right)$$

$$\boxed{f(p) = \frac{2i}{p^2}\left(\sin ap - ap\cos ap \right)}$$

Q.2 . Final the complex Fourier transform of $f(x) = e^{-ax^2}$, $(a > 0)$. Hence find the Fourier transform of

(i) $F(x) = e^{-x^2}$ **(ii) $F(x) = e^{-x^2/2}$** **(AKTU-2017)**

$$F(x) = e^{-ax^2}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-ax^2} \times e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-ax^2 + ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x^2 - \frac{ipx}{a}\right)}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[x^2 - \frac{ip}{a}x + \left(\frac{ip}{2a}\right)^2 - \left(\frac{ip}{2a}\right)^2\right]}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[\left(x - \frac{ip}{2a}\right)^2 - \frac{i^2p^2}{4a^2}\right]}$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[\left(x - \frac{ip}{2a}\right)^2 + \frac{p^2}{4a^2}\right]}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{ip}{2a}\right)^2 - \frac{ap^2}{4a^2}} dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{ip}{2a}\right)^2} \times e^{\frac{-p^2}{4a}} dx$$

$$f(p) = e^{-p^2/4a} \int_{-\infty}^{\infty} e^{-\left[\sqrt{a}\left(x - \frac{ip}{2a}\right)\right]^2} dx$$

Put $\sqrt{a}\left(x - \frac{ip}{2a}\right) = t$

$$\sqrt{a}\, dx = dt$$

$$dx = \frac{dt}{\sqrt{a}}$$

$$f(p) = e^{\frac{-p^2}{4a}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{a}}$$

$$f(p) = \frac{e^{\frac{-p^2}{4a}}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$\boxed{f(p) = \frac{e^{\frac{-p^2}{4a}} \times \sqrt{\pi}}{\sqrt{a}}} \quad \checkmark \quad \text{---(1)}$$

(i) $F(x) = e^{-x^2}$

Here $a = 1$

Put $a =$ in (1)

$$f(p) = e^{-\frac{p^2}{4}} \times \sqrt{\pi}$$

(ii) $F(x) = e^{-\frac{x^2}{2}}$

Here $a = \frac{1}{2}$

Put $a = \frac{1}{2}$ in (1)

$$f(p) = e^{-\frac{p^2}{2}} \times \sqrt{\pi} \times \sqrt{2}$$

$$f(p) = e^{-\frac{p^2}{2}} \times \sqrt{2\pi}$$

Q.3 . Find the Fourier transform of

$$F(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$$

(UPTU-2016, 2014 & 2010)

Hence evaluate (i) $\int_{-\infty}^{\infty} \left(\frac{\sin as \cos xs}{s}\right) ds.$ (ii) $\int_{0}^{\infty} \left(\frac{\sin s}{s}\right) ds.$

$$F(x) = \begin{cases} 1 & -a < x < a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = \int_{-\infty}^{-a} F(x) e^{ipx} dx + \int_{-a}^{a} F(x) e^{ipx} dx + \int_{a}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = 0 + \int_{-a}^{a} 1 \times e^{ipx} dx + 0$$

$$f(p) = \int_{-a}^{a} e^{ipx} dx$$

$$f(p) = \left(\frac{e^{ipx}}{ip}\right)_{-a}^{a}$$

$$f(p) = \frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip}$$

$$f(p) = \frac{1}{ip}\left(e^{ipa} - e^{-ipa}\right)$$

$$f(p) = \frac{2i}{ip}\left(\frac{e^{ipa} - e^{-ipa}}{2i}\right)$$

$$\boxed{f(p) = \frac{2}{p}\sin ap}$$

(i) By Inverse Fourier transform

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin ap}{p}\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{p}\sin ap\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{p}\sin ap\,(\cos px - i\sin px)\,dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin ap\cos px}{p} - i\frac{\sin ap\sin px}{p}\right)dp$$

even odd

By property of Definite integral

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap\cos px}{p}\,dp$$

Put $p = s$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as\cos sx}{s}\,ds$$

$$\int_{-\infty}^{\infty} \frac{\sin as\cos sx}{s}\,ds = \pi F(x)$$

$$\int_{-\infty}^{\infty} \frac{\sin as\cos sx}{s}\,ds = \pi F(x)$$

$$\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \pi \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases}$$

(11) Put $a = 1$ and $x = 0$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi F(0)$$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi \cdot 1$$

even

By the property of Definite integral

$$2\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

$$\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

Q.4 Find the Fourier transform of $\begin{cases} 1-x^2, & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$ and use it to Evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$.

(UPTU-2015)

$$F(x)=\begin{cases} 1-x^2 & -1<x<1 \\ 0 & x>1 \text{ or } x<-1 \end{cases}$$

By Fourier transform

$$f(p)=\int_{-\infty}^{\infty} F(x)\,e^{ipx}\,dx$$

$$f(p)=\int_{-\infty}^{-1} F(x)e^{ipx}dx+\int_{-1}^{1} F(x)e^{ipx}dx+\int_{1}^{\infty} F(x)e^{ipx}dx$$

$$f(p)=0+\int_{-1}^{1}(1-x^2)e^{ipx}dx+0$$

$$f(p)=\int_{-1}^{1}(1-x^2)e^{ipx}dx$$

I II

$$f(p)=\left[(1-x^2)\left(\frac{e^{ipx}}{ip}\right)-(-2x)\left(\frac{e^{ipx}}{i^2p^2}\right)+(-2)\left(\frac{e^{ipx}}{i^3p^3}\right)\right]_{-1}^{1}$$

$$f(p) = \left[(1-n^2)\frac{e^{ipn}}{ip} - \frac{2ne^{ipn}}{p^2} + \frac{2e^{ipn}}{ip^3} \right]_{-1}^{1}$$

$$f(p) = \left[0 - \frac{2e^{ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \left(0 + \frac{2e^{-ip}}{p^2} + \frac{2e^{-ip}}{ip^3} \right) \right]$$

$$f(p) = -\frac{2e^{ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \frac{2e^{-ip}}{p^2} - \frac{2e^{-ip}}{ip^3}$$

$$f(p) = -\frac{2}{p^2}\left(e^{ip} + e^{-ip} \right) + \frac{2}{ip^3}\left(e^{-ip} - e^{-ip} \right)$$

$$f(p) = -\frac{4}{p^2}\left(\frac{e^{ip}+e^{-ip}}{2}\right) + \frac{4}{p^3}\left(\frac{e^{ip}-e^{-ip}}{2i}\right)$$

$$f(p) = -\frac{4}{p^2}\cos p + \frac{4}{p^3}\sin p$$

$$\boxed{f(p) = \frac{4}{p^3}\left(\sin p - p\cos p\right)}$$

By inverse Fourier transform

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^3}\left(\sin p - p\cos p\right) e^{-ipx}\, dp$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4^2}{p^3}(\sin p - p\cos p)(\cos px - i\sin px)\,dp$$

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\underbrace{\frac{\sin p\cos px}{p^3}}_{\text{even}} \underbrace{- i\frac{\sin p\sin px}{p^3}}_{\text{odd}} \underbrace{- \frac{p\cos p\cos px}{p^3}}_{\text{even}} + \underbrace{i\frac{p\cos p\sin px}{p^3}}_{\text{odd}}\right)dp$$

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p\cos px}{p^3} - \frac{p\cos p\cos px}{p^3}\right)dp$$

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos px\,dx$$

Put $x = \frac{1}{2}$

$$F\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos\frac{p}{2}\,dp$$

Replace p by x

$$F\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx = \frac{\pi}{2}F\left(\frac{1}{2}\right)$$

$$\int_{-\infty}^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{\pi}{2}F\left(\frac{1}{2}\right) \quad \text{— (1)}$$

$$F(x) = 1 - x^2$$

$$F\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2$$

$$F\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} \left(\frac{x\cos x - \sin x}{x^3} \right) \cos\frac{x}{2}\, dx = -\frac{3\pi}{8}$$

DPP- 11

Topic : *Complex Fourier transform*

Q.1. *Find the Fourier Transform of* $F(x) = \begin{cases} x^2, & if\ |x| < a \\ 0, & if\ |x| > a \end{cases}$

Q.2. *Find the Fourier Transform of* $f(x) = e^{-|x|}$

Q.3. *Find the Fourier transform of* $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. *hence avaluate* $\int_0^\infty \frac{\sin x}{x} dx$. *(GBTU-2011)*

Q. 4. *Find the Fourier transform of Block function* $f(t)$ *of height 1 and duration a defined by* *(AKTU-2016)*

$$f(t) = \begin{cases} 1, & for\ |t| \le \frac{a}{2} \\ 0, & otherwise \end{cases}$$

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