DPP-10 Mathr-IV Unit-2

Topic - Two Dimensional Heat Equation

Q-1: Solve the Laplace equation $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ in a rectangle in the xy-plane with $u(x,0) = 0$, $u(x,b) = 0$, $u(0,y) = 0$ and $u(a,y) = f(y)$ parallel to y-axis.

Sol- Let $u(x,y)$ be the steady state temp given by Laplace equation

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \quad -(1)$$

Using the Boundry conditions

$u(x,0) = 0$ BC-1

$u(x,b) = 0$ BC-2

$u(0,y) = 0$ BC-3

$u(a,y) = f(y)$ BC-4

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Most suitable solution of eq (1)

When $k = p^2$

$$u(x,y) = \left(C_1 e^{px} + C_2 e^{-px}\right)\left(C_3 \cos py + C_4 \sin py\right)$$

Apply BC-1,

$$0 = \left(C_1 e^{px} + C_2 e^{-px}\right) \times C_3$$

$$\boxed{C_3 = 0}$$

Updated solution,

$$u(x,y) = \left(C_1 e^{px} + C_2 e^{-px}\right) \times C_4 \sin py$$

Apply BC-2;

$$0 = \left(C_1 e^{px} + C_2 e^{-px}\right) \times C_4 \sin pb$$

$$\sin pb = 0$$

$$pb = n\pi$$

$$\boxed{p = \frac{n\pi}{b}}$$

Updated solution,

$$u(x,y) = \left(C_1 e^{px} + C_2 e^{-px}\right) \times C_4 \sin \frac{n\pi}{b} y$$

$$u(x,y) = \left(C_1 e^{\frac{n\pi x}{b}} + C_2 e^{-\frac{n\pi x}{b}}\right) \times C_4 \sin \frac{n\pi}{b} y$$

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Apply BC-3,

$$0 = (C_1 + C_2) \times C_4 \sin \frac{n\pi}{b} y$$

$$C_1 + C_2 = 0$$

$$\boxed{C_1 = -C_2}$$

Updated solution;

$$u(x,y) = 2C_1C_4 \sin \frac{n\pi}{b} y \left(\frac{e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}}{2} \right)$$

$$u(x,y) = 2C_1C_4 \sin \frac{n\pi}{b} y \sinh \frac{n\pi x}{b}$$

Most general solution :-

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{b} y \sinh \frac{n\pi x}{b}$$

Apply BC-4,

$$f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{b} y \sinh \frac{n\pi a}{b}$$

Apply half range sine formula

$$\left(\sinh \frac{n\pi a}{b}\right) b_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi}{b} y \, dy$$

$$b_n = \frac{2}{b\left(\sinh \frac{n\pi a}{b}\right)} \int_0^b f(y) \sin \frac{n\pi}{b} y \, dy$$

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$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b}$$

Where $b_n = \dfrac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\frac{n\pi y}{b}\, dy$

Q-2: Solve $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ subject to the conditions

$u(0,y)$, $u(a,y) = 0$, $u(x,0) = 0$ and $u(x,b) = x$

Sol: Let $u(x,y)$ be the steady state temp given by Laplace equation

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \quad -(1)$$

Using the boundry conditions,

$u(0,y) = 0$ BC-1
$u(a,y) = 0$ BC-2
$u(x,0) = 0$ BC-3
$u(x,b) = 0$ BC-4

Most suitable solution:

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

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Apply BC-1,

$$0 = C_1 \left(C_3 e^{py} + C_4 e^{-py}\right)$$

$$\boxed{C_1 = 0}$$

Updated solution,

$$u(x,y) = C_2 \sin px \left(C_3 e^{py} + C_4 e^{-py}\right)$$

Apply BC-2,

$$0 = C_2 \sin pa \left(C_3 e^{py} + C_4 e^{-py}\right)$$

$$\sin pa = 0$$

$$pa = n\pi$$

$$\boxed{p = \frac{n\pi}{a}}$$

Updated solution,

$$u(x,y) = C_2 \sin \frac{n\pi x}{a} \left(C_3 e^{\frac{n\pi y}{a}} + C_4 e^{-\frac{n\pi y}{a}}\right)$$

Apply BC-3,

$$0 = C_2 \sin \frac{n\pi x}{a} \left(C_3 + C_4\right)$$

$$C_3 + C_4 = 0$$

$$\boxed{C_3 = -C_4}$$

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Updated solution,

$$u(x,y) = C_2 C_3 \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}} \right)$$

Most general solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}} \right)$$

Apply BC-4,

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}} \right)$$

$$x = \sum_{n=1}^{\infty} b_n' \sin \frac{n\pi x}{a}$$

Apply Half range sine series

$$b_n' = \frac{2}{a} \int_0^a x \sin \frac{n\pi x}{a} \, dx$$

(I) (II)

$$b_n' = \frac{2}{a} \left[\left(-x \frac{\cos \frac{n\pi x}{a}}{n\pi/a} \right)_0^a - \int_0^a \frac{-\cos \frac{n\pi x}{a}}{n\pi/a} \right]$$

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$$b_n' = \frac{2}{a}\left[\left(\frac{-x\cos\frac{n\pi x}{a}}{n\pi/a}\right)_0^a + \left(\frac{\sin\frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right)_0^a\right]$$

$$b_n' = \frac{2}{a}\left[\left(\frac{a\cos n\pi - 0}{n\pi/a}\right) + \left(\frac{\sin n\pi - 0}{\frac{n^2\pi^2}{a^2}}\right)\right]^0$$

$$b_n' = \frac{2}{a} \times \frac{-a^2\cos n\pi}{n\pi}$$

$$b_n' = -\frac{2a}{\pi}\frac{\cos n\pi}{n}$$

$$b_n\left(e^{\frac{n\pi b}{a}} - e^{\frac{-n\pi b}{a}}\right) = -\frac{2a}{\pi}\frac{\cos n\pi}{n}$$

$$2b_n\left(\frac{e^{\frac{n\pi b}{a}} - e^{\frac{-n\pi b}{a}}}{2}\right) = -\frac{2a}{\pi}\frac{\cos n\pi}{n}$$

$$2b_n \sinh\left(\frac{n\pi b}{a}\right) = -\frac{2a}{\pi}\frac{\cos n\pi}{n}$$

$$b_n = \frac{-\frac{2a}{\pi}\frac{\cos n\pi}{n}}{2\sinh\left(\frac{n\pi b}{a}\right)}$$

Put value of b_n in most general solution

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$$u(x,y) = \sum_{n=1}^{\infty} \frac{-2a}{n\pi} \frac{\cos n\pi}{\sinh\left(\frac{n\pi b}{a}\right)} \left(\frac{e^{\frac{n\pi y}{a}} - e^{\frac{-n\pi y}{a}}}{2}\right) \sin\frac{n\pi x}{a}$$

$$u(x,y) = \frac{-2a}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n \sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$u(x,y) = \frac{-2a}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)}{n \sinh\left(\frac{n\pi b}{a}\right)}$$

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