ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (PART-1
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

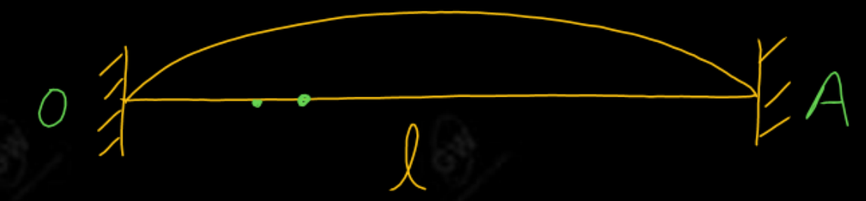
One Dimensional wave equation

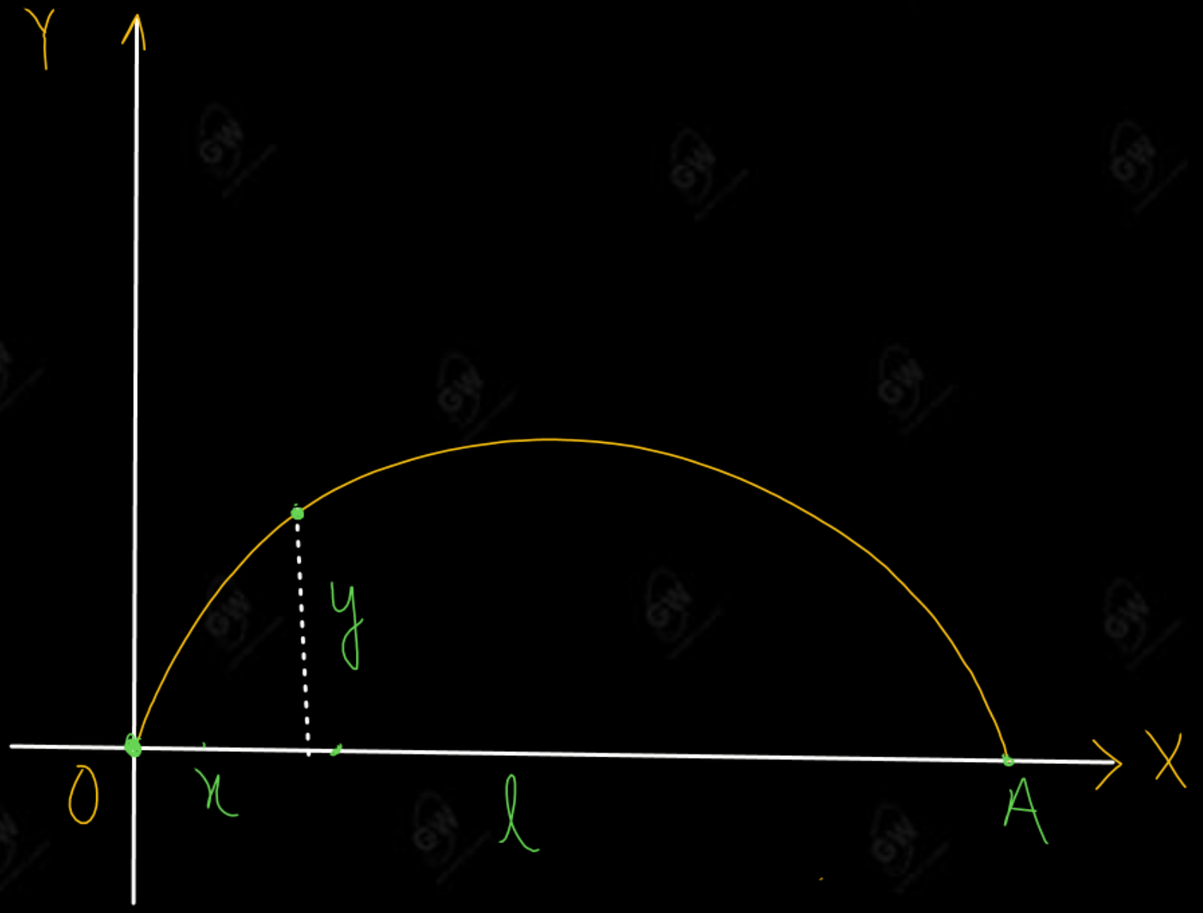
OR

Equation of vibrating string

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Vibrations of a stretched string





Boundary conditions

(i) At $x = 0$, $y = 0$

(ii) At $x = l$, $y = 0$

Initial condition

At $t = 0$, $y = f(x)$

At $t = 0$, $\frac{\partial y}{\partial t} = 0$

Q.1. Solve one dimensional wave equation by using Method of separation of variables.

one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

Let $y = X(x)T(t)$

$$y = XT \quad \text{— (2)}$$

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2} = XT''$$

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{\partial x^2} = TX''$$

Put these values in (1)

$$XT'' = c^2 TX''$$

$$XT'' = c^2 TX''$$

Divide by XT

$$\frac{XT''}{XT} = c^2 \frac{TX''}{XT}$$

$$\frac{T''}{T} = c^2 \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{—— (3)}$$

$$\frac{1}{c^2}\frac{T''}{T} = K$$

$$T'' = c^2 K T$$

$$T'' - c^2 K T = 0$$

$$D^2T - c^2 K T = 0$$

$$(D^2 - c^2K)T = 0 \quad \text{—— (4)}$$

CASE-1 : $K = 0$

From (3)

$$D^2X = 0$$

Aux Eqn

$$D^2 = 0$$

$$D = 0, 0$$

$$X = C_1 + C_2 x$$

From (4)

$$D^2T = 0$$

Aux. Eqn

$$D^2 = 0$$

$$D = 0, 0$$

$$T = C_3 + C_4 t$$

Put X and T in (2)

$$y = (C_1 + C_2 x)(C_3 + C_4 t) \quad \text{—— (5)}$$

CASE-2: $K = p^2$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D = \pm p$$

$$X = C_1 e^{px} + C_2 e^{-px}$$

From (4)

$$(D^2 - c^2 p^2)T = 0$$

$$D^2 - c^2 p^2 = 0$$

$$D = \pm cp$$

$$T = C_3 e^{cpt} + C_4 e^{-cpt}$$

Put X and T in (2)

$$y = \left(C_1 e^{px} + C_2 e^{-px}\right)\left(C_3 e^{cpt} + C_4 e^{-cpt}\right)$$

—— (6)

CASE -3 : $K = -p^2$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$X = C_1 \cos px + C_2 \sin px$$

From (4)

$$(D^2 + c^2p^2)T = 0$$

$$D^2 + c^2p^2 = 0$$

$$D = \sqrt{-c^2p^2}$$

$$D = \pm icp$$

$$D = 0 \pm icp$$

$$T = C_3 \cos cpt + C_4 \sin cpt$$

Put X and T in (2)

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

(7)

Most suitable solution

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$ — (8)

Apply Boundary condition

(i) At $x = 0$, $y = 0$ ✓

(ii) At $x = l$, $y = 0$

$$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$$

$$\boxed{C_1 = 0}$$

Updated solution

$$y = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)$$ — (9)

$$0 = C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$$

$$\sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Initial conditions

(i) At $t = 0$, $y = f(x)$ ✓✓

(ii) At $t = 0$, $\frac{\partial y}{\partial t} = 0$ ✓✓

Updated solution

$$y = C_2 \sin\frac{n\pi}{l}x\left(C_3\cos\left(\frac{n\pi}{l}t\right) + C_4\sin\left(\frac{n\pi}{l}t\right)\right)$$

└ (10)

$$\frac{\partial y}{\partial t} = C_2 \sin\frac{n\pi x}{l}\left(-C_3\sin\frac{cn\pi t}{l}\times\frac{cn\pi}{l} + C_4\cos\frac{cn\pi t}{l}\times\frac{cn\pi}{l}\right)$$

$$0 = C_2 \sin\frac{n\pi x}{l}\left(0 + C_4\times\frac{cn\pi}{l}\right)$$

$$0 = C_2 C_4 \frac{cn\pi}{l}\sin\frac{n\pi x}{l}$$

$$\Rightarrow C_4 = 0$$

Updated solution

$$y = C_2 \sin\frac{n\pi x}{l} C_3\cos\frac{cn\pi t}{l}$$

$$y = (C_2 C_3) \sin\frac{n\pi x}{l} \cos\frac{nc\pi t}{l}$$

Most General solution

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{nc\pi t}{l}$$

At $t = 0$, $y = f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$ ✓✓

$b_n = ?$

- Comparison
- Fourier series (Half range sine series)

Engineering Mathematics by Gulshan Sir

DPP- 3

Topic : Solution of one Dimensional Wave Equation

Q.1. ***Derive one dimensional wave equation.***

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Thank You

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