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DPP-6 Maths-IV Unit-2

Topic- One Dimenssional Heat Equation

Q-1: A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled at 0°C and are kept at that temperature. Find the temperature function $u(x,t)$.

Sol: Boundry conditions

At $x=0$, $u=0$ $\Rightarrow u(0,t)=0°C$ BC-1

At $x=l$, $u=0$ $\Rightarrow u(l,t)=0°C$ BC-2

$u_s(x) = a+bx$

Apply BC-1

$0 = a+0$
$a = 0$

$u_s(x) = bx$
$0 = bl$
$b = 0$

$\boxed{u_s(x)=0}$

$u(x,t) = u_s(x) + u_t(x,t)$

↓ Steady State ↓ Transient state

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$$u(x,t) = u_t(x,t)$$

$$\boxed{u(x,t) = (C_1 \cos px + C_2 \sin px)\ C_3 e^{-c^2p^2t}}$$

Apply BC-1

$$0 = C_1 C_3 e^{-c^2p^2t}$$

$$\boxed{C_1 = 0}$$

Updated solution,

$$u(x,t) = C_2 \sin px \times C_3 e^{-c^2p^2t}$$

Apply BC-2;

$$0 = C_2 \sin pl \times C_3 e^{-c^2p^2t}$$

$$\sin pl = 0$$
$$pl = n\pi$$
$$p = \frac{n\pi}{l}$$

Updated solution-

$$u(x,t) = C_2 \sin \frac{n\pi}{l} x \times C_3 e^{-c^2 \frac{n^2\pi^2}{l^2} t}$$

$$u(x,t) = C_2C_3 \sin \frac{n\pi}{l} x \times e^{-c^2 \frac{n^2\pi^2}{l^2} t}$$

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Most general solution,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

Apply IC, $t = 0$

$$u_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad -(11)$$

Using Half range sine series,

$$b_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\, dx$$

$$b_n = \frac{2u_0}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l$$

$$b_n = \frac{2u_0}{\cancel{l}} \times \frac{\cancel{l}}{n\pi}\left[-\cos n\pi + 1\right]$$

$$b_n = \frac{2u_0}{n\pi}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{4u_0}{n\pi} & \text{when } n \text{ is odd} \end{cases}$$

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Put b_n in (11).

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

$$= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\frac{(2n-1)\pi x}{l}\, e^{-\frac{c^2 (2n-1)^2 \pi^2 t}{l^2}}$$

Ans

Q-2: Determine the solution of one-dimensional heat equation $\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2}$ where boundry conditions are $u(0,t) = 0 = u(l,t)$ and the initial condition $u(x,0) = x$: l being the length of bar.

Sol:- $$\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2} \quad -(1)$$

length of rod = l m

Boundry conditions, At $x = 0$, $u = 0$ BC-1
At $x = l$, $u = 0$ BC-2

Initial condition, At $t = 0$, $u = f(x) = x$

$$u(x,t) = u_s(x) + u_t(x,t)$$

$$u_s(x) = 0$$

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$u(x,t) = u_t(x,t)$

Most suitable solution,

$$u(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-c^2p^2t}$$

Apply BC-1;

$$0 = C_1 C_3 e^{-c^2p^2t}$$

$$\boxed{C_1 = 0}$$

Updated solution $\Rightarrow$ $u(x,t) = C_2 \sin px \times C_3 e^{-c^2p^2t}$

Apply BC-2

$$0 = C_2 \sin pl \times C_3 e^{-c^2p^2t}$$

$$\sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Updated solution $\Rightarrow$

$$u(x,t) = C_2 \sin \frac{n\pi}{l} x \times C_3 e^{-c^2p^2t}$$

Most General sol,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times e^{-c^2p^2t} \quad -(11)$$

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Apply IC :-

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Using Half-range sine series

$$b_n = \frac{2}{l} \int_0^l x \times \sin \frac{n\pi x}{l} \, dx$$

I II

$$b_n = \frac{2}{l} \left[x \times \frac{-\cos \frac{n\pi x}{l}}{n\pi/l} - \int \frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right]_0^l$$

$$b_n = \frac{2}{l} \left[x \times \frac{-\cos \frac{n\pi x}{l}}{n\pi/l} + \frac{\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2} \right]_0^l$$

$$b_n = \frac{2}{l} \left[\frac{l \times -\cos n\pi}{n\pi/l} + \frac{\overset{0}{\cancel{\sin n\pi}}}{n^2\pi^2/l^2} \right]$$

$$b_n = \frac{2}{\cancel{l}} \times \cancel{l} \times \frac{-\cos n\pi}{n\pi/l}$$

$$b_n = \frac{2l}{n\pi} (-\cos n\pi)$$

Put b_n in eq (11)

$$u(x,t) = -\frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin \frac{n\pi x}{l} \, e^{-\left(\frac{c^2 n^2 \pi^2 t}{l^2}\right)}$$

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