ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Two Dimensional Heat Equation (Laplace Equation) Part-1
- PYQ
- DPP

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Application of partial differential equations

1. Solution of one Dimensional Wave Equation 3 L
2. Solution of one Dimensional Heat Equation 3 L
3. Solution of two Dimensional heat Equation 2 L

Method of Separation of Variables 2 L

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Two Dimensional Heat Equation (Laplace Equation)

(i) Two Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(ii) Two Dimensional Heat Equation in steady state (Laplace Equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\hookrightarrow \frac{\partial u}{\partial t} = 0$

Known as Laplace equation

Solution of Laplace equation in two dimension by using Separation of Variables Method.

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} (1)$$

Let $u(x, y) = XY$ —— (2)

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{d^2X}{dx^2} = YX''$$

$$\frac{\partial^2 u}{\partial y^2} = X\frac{d^2Y}{dy^2} = XY''$$

Put these values in (1)

$$YX'' + XY'' = 0$$

Divide by XY

$$\frac{YX''}{XY} + \frac{XY''}{XY} = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{——} (3)$$

$$\frac{-Y''}{Y} = K$$

$$-Y'' = kY$$

$$Y'' + kY = 0$$

$$D^2Y + kY = 0$$

$$(D^2 + k)Y = 0 \quad \text{—(4)}$$

CASE–1 : When $k = 0$

$$(D^2 - k)X = 0$$

$$D^2X = 0$$

$$D^2 = 0$$

$$D = 0, 0$$

$$X = c_1 + c_2x$$

$$(D^2 + k)Y = 0$$

$$D^2Y = 0$$

$$D^2 = 0$$

$$D = 0, 0$$

$$Y = c_3 + c_4y$$

Put x and y in (2)

$$u(x, y) = (c_1 + c_2x)(c_3 + c_4y)$$

(5)

CASE-2: When $k = p^2$

$$(D^2 - k)X = 0$$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D^2 = p^2$$

$$D = \pm p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$(D^2 + k)Y = 0$$

$$(D^2 + p^2)Y = 0$$

$$D^2 + p^2 = 0$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$Y = c_3 \cos py + c_4 \sin py$$

Put X and Y in (2)

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

(6)

CASE-3: When $k = -p^2$

$$(D^2 - k)X = 0$$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D^2 = -p^2$$

$$D = \sqrt{-p^2}$$

$$D = 0 \pm ip$$

$$X = c_1 \cos px + c_2 \sin px$$

$$(D^2 + k)Y = 0$$

$$(D^2 - p^2)Y = 0$$

$$D^2 - p^2 = 0$$

$$D^2 = p^2$$

$$D = \pm p$$

$$Y = c_3 e^{py} + c_4 e^{-py}$$

Put these values in (2)

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

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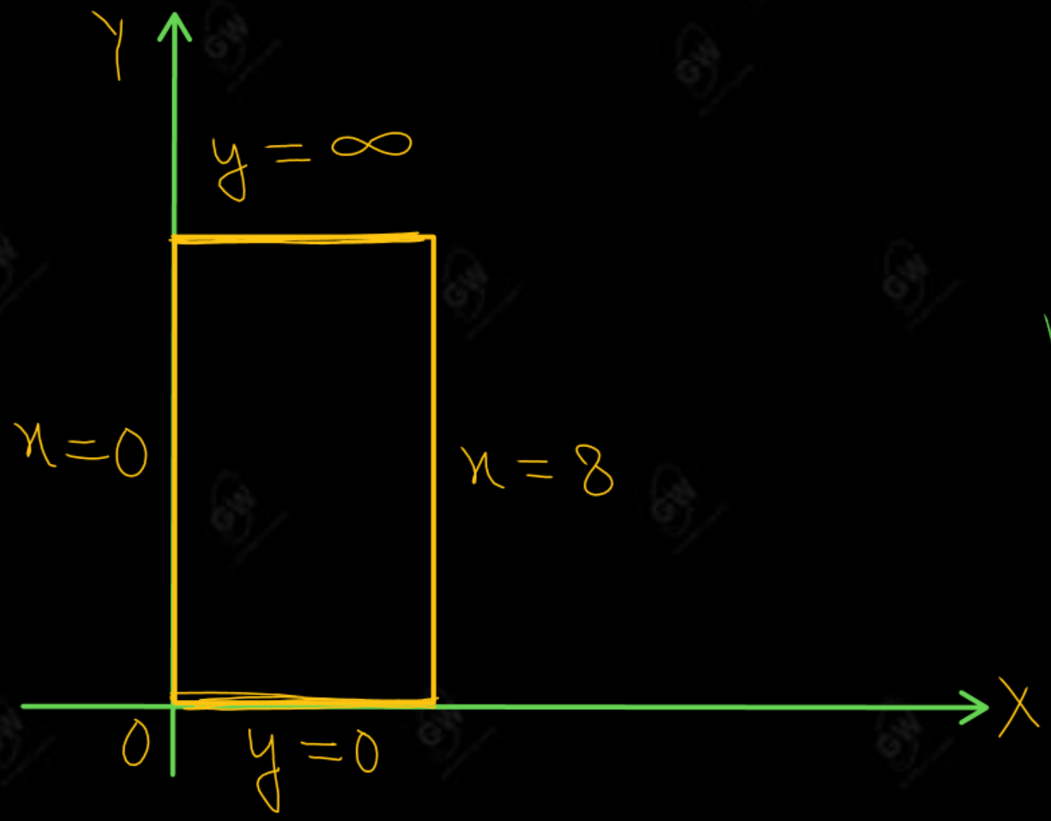
Solution of Laplace Equation

$$u(x,y) = (C_1 + C_2 x)(C_3 + C_4 y) \quad \text{—— (5)} \qquad \times \qquad k = 0$$

$$u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad \text{— (6)} \qquad k = p^2$$

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (7)} \qquad k = -p^2$$

Q. 1: A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}, 0 < x < 8$ *while the two long edges* $x = 0$ *and* $x = 8$ *as well as the other short edge are kept at* 0^0 *C, show that the steady state temperature at any point of the plate is given by* $u(x, y) = 100\, e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}$.



Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} (1)$$

where $u(x, y)$ is steady state temp of plate

Under the Boundary condition

$u(0,y) = 0$ BC-1

$u(8,y) = 0$ BC-2

$u(x,\infty) = 0 \Rightarrow \lim_{y\to\infty} u(x,y) = 0$ BC-3

$u(x,0) = 100 \sin\frac{\pi x}{8}$ BC-4

solution of Laplace equation

$$u(x,y) = (C_1 + C_2 x)(C_3 + C_4 y) \quad -(5)$$

$$u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad -(6)$$

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad -(7)$$

Most suitable solution

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad -(8)$$

Apply BC-1

$$0 = C_1(C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$u(x,y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (9)}$$

Apply BC-2

$$0 = C_2 \sin 8p\,(C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow \sin 8p = 0$$

$$\sin 8p = n\pi$$

$$8p = n\pi$$

$$p = \frac{n\pi}{8}$$

Updated solution

$$u(x,y) = C_2 \sin \frac{n\pi}{8} x \left(C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}}\right) \quad \text{— (10)}$$

Apply BC-3

$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} C_2 \sin\frac{n\pi x}{8}\left(C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}}\right)$$

$$0 = C_2 \sin\frac{n\pi x}{8}\left[\lim_{y\to\infty} C_3 e^{\frac{n\pi y}{8}} + \lim_{y\to\infty} C_4 e^{\frac{-n\pi y}{8}}\right]$$

$$\Rightarrow C_3 = 0$$

Updated solution

$$u(x,y) = C_2 \sin\frac{n\pi x}{8} \times C_4 e^{-\frac{n\pi y}{8}} \quad \text{— (11)}$$

Most general solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8} e^{-\frac{n\pi y}{8}}$$

Apply BC-4

$$100 \sin\frac{\pi x}{8} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8}$$

$$100 \sin\frac{\pi x}{8} = b_1 \sin\frac{\pi x}{8} + b_2 \sin\frac{2\pi x}{8} + \cdots$$

on comparison

$$b_1 = 100$$

$$n = 1$$

$$u(x,y) = b_1 \sin\frac{\pi x}{8} e^{-\frac{\pi y}{8}}$$

$$u(x,y) = 100 \sin\frac{\pi x}{8} e^{-\frac{\pi y}{8}}$$

$$\boxed{u(x,y) = 100 e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}}$$

Q.1. **Solve temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by partial differential equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < \pi, 0 < y < \pi$$ **, which satisfies the conditions :**

$$u(0,y) = u(\pi, y) = u(x, \pi) = 0 \text{ and } u(x, 0) = sin^2 x$$ (U.K.T.U. 2011)

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