DPP-8 Maths-IV Unit-2

Topic- One Dimensional Heat Equation

Q-1 (i) An insulated rod length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.

ii) Find also the temperature if the change consists of raising of the temperature of A to 20°C and reducing that of B to 80°C.

Sol-i) $$\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2} \quad -(1)$$

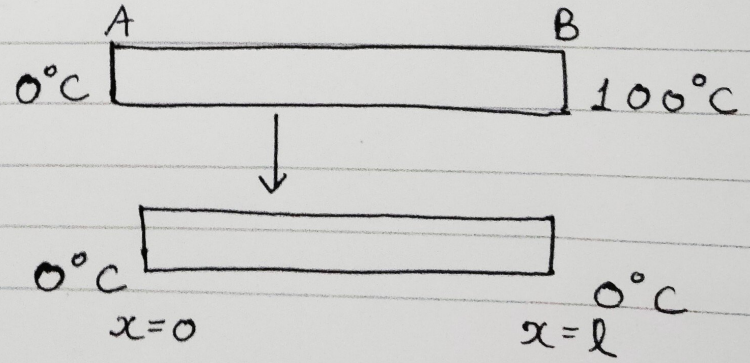
length of bar $(l) = l$ cm

Boundry conditions,

At $x = 0$, $u = 0°$ BC-1
At $x = l$, $u = 0°$ BC-2

Initial condition,

$$u = T_A + \left(\frac{T_B - T_A}{l}\right) x$$



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$$u = 0 + \frac{100x}{l}$$

At $t = 0$, $u = \frac{100x}{l}$ — IC

Complete solution of (1)

$$u(x,t) = u_s(x) + u_s(x,t) \quad — (8)$$

Steady state solution -

$$u_s(x) = 0 + 0$$
$$= 0$$

$$\boxed{u_s(x) = 0}$$

$$u(x,t) = u_s(x,t)$$

$$u(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-c^2p^2t}$$

Apply BC-1;

$$0 = C_1 C_3 e^{-c^2p^2t}$$

$$\boxed{C_1 = 0}$$

Updated solution,

$$u(x,t) = C_2 \sin px \times C_3 e^{-c^2p^2t}$$

Apply BC-2

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$$0 = C_2 \sin pl \times C_3 e^{-c^2p^2t}$$

$$\sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Updated solution :- $u(x,t) = C_2 \sin \frac{n\pi}{l} x \times C_3 e^{-c^2 \frac{n^2\pi^2}{l^2} t}$

Most general solution :-

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \times e^{-c^2 \frac{n^2\pi^2}{l^2} t} \quad -(12)$$

Apply Initial condition -

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

Apply Half range sine series,

$$b_n = \frac{2}{l} \int_0^l \underset{I}{\frac{100x}{l}} \underset{II}{\sin \frac{n\pi}{l} x} \, dx$$

$$b_n = \frac{200}{l^2} \left[\left\{ x \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right) \right\}_0^l - \int_0^l 1 \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l} \right) dx \right]$$

$$b_n = \frac{200}{l^2} \left[\frac{-l^2}{n\pi} \cos n\pi + \frac{l}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{n\pi/l} \right)_0^l \right]$$

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$$b_n = \frac{200}{l^2}\left[\frac{-l^2}{n\pi}\cos n\pi + \frac{l^2}{n^2\pi^2}\left(\sin^{0} n\pi - \sin^{0} 0\right)\right]$$

$$b_n = \frac{200}{l^2} \times \frac{-l^2}{n\pi}\cos n\pi$$

$$b_n = \frac{-200}{n\pi}(-1)^n$$

Put value of b_n in (12)

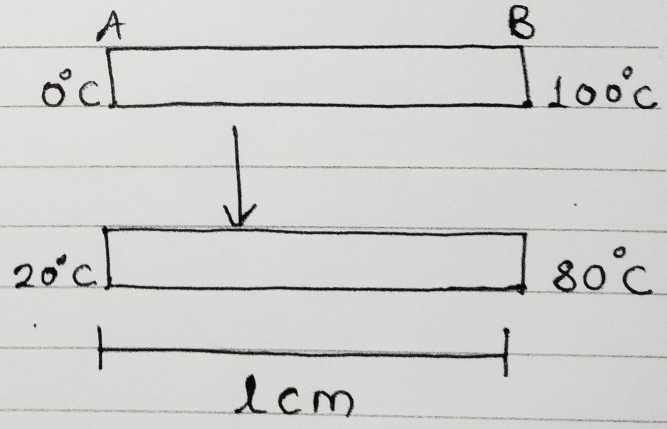
$$u(x,t) = \frac{-200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

Answer

ii) Boundry conditions,

At $x = 0$, $u = 20^\circ C$ BC-1

At $x = l$, $u = 80^\circ C$ BC-2



Initial condition

$$u = T_A + \left(\frac{T_B - T_A}{l}\right)x$$

$$u = 0 + \frac{100}{l}x$$

$$u = \frac{100x}{l}$$

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Complete solution of (1)

$$u(x,t) = u_s(x) + u_s(x,t)$$

$$u(x,t) = a + bx + (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2p^2t} \quad -(8)$$

Steady state solution,

$$u_s(x) = T_A + \left(\frac{T_B - T_A}{l}\right) x$$

$$= 20 + \left(\frac{80-20}{l}\right) x$$

$$\boxed{u_s(x) = 20 + \frac{60x}{l}}$$

Put in eq -(8)

$$u(x,t) = 20 + \frac{60x}{l} + (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2p^2t} \quad -(9)$$

Apply BC-1;

$$20 = 20 + C_1 C_3 e^{-c^2p^2t}$$

$$0 = C_1 C_3 e^{-c^2p^2t}$$

$$\boxed{C_1 = 0}$$

Updated solution ⇒

$$u(x,t) = 20 + \frac{60x}{l} + C_2 \sin px \times C_3 e^{-c^2p^2t} \quad -(10)$$

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Apply BC-2;

$$80 = 20 + 60 + C_2 \sin pl \times C_3 e^{-c^2p^2t}$$

$$80 - 80 = C_2 \sin pl \times C_3 e^{-c^2p^2t}$$

$$\sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Updated solution :-

$$u(x,t) = 20 + \frac{60x}{l} + C_2 \sin \frac{n\pi}{l} x \times C_3 e^{-c^2 \frac{n^2\pi^2}{l^2} t} \quad -(11)$$

Most general solution,

$$u(x,t) = 20 + \frac{60x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times e^{-c^2 \frac{n^2\pi^2}{l^2} t} \quad -(12)$$

Apply Initial conditions,

$$\frac{100x}{l} = 20 + \frac{60x}{l} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times 1$$

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Apply Half range sine series,

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$$b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{l}\left[\left\{\left(\frac{40x}{l} - 20\right)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)\right\}_0^l - \int_0^l \frac{40}{l}\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)dx\right]$$

$$b_n = \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{20l}{n\pi} + \frac{40}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right)_0^l\right]$$

$$b_n = \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{20l}{n\pi} + \frac{40l}{n^2\pi^2}\left(\sin n\pi - \sin 0\right)\right]$$

$$b_n = -\frac{40}{n\pi}\cos n\pi - \frac{40}{n\pi}$$

$$b_n = -\frac{40}{n\pi}\left(1 + \cos n\pi\right)$$

$$b_n = -\frac{40}{n\pi}\left(1 + (-1)^n\right)$$

→ 0, when n is odd

→ $-\frac{80}{n\pi}$, when n is even

$$u(x,t) = 20 + \frac{60x}{l} - \frac{80}{\pi}\sum_{\substack{n=2,4,..\infty \\ (\text{even})}} \frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

$$u(x,t) = 20 + \frac{60x}{l} - \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{m}\sin\frac{2m\pi x}{l}\, e^{-\frac{4c^2m^2\pi^2 t}{l^2}}$$

{For even Takes $n = 2m$

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