Engineering Mathematics

UNIT : Application of Partial Differential Equation and Fourier Transform

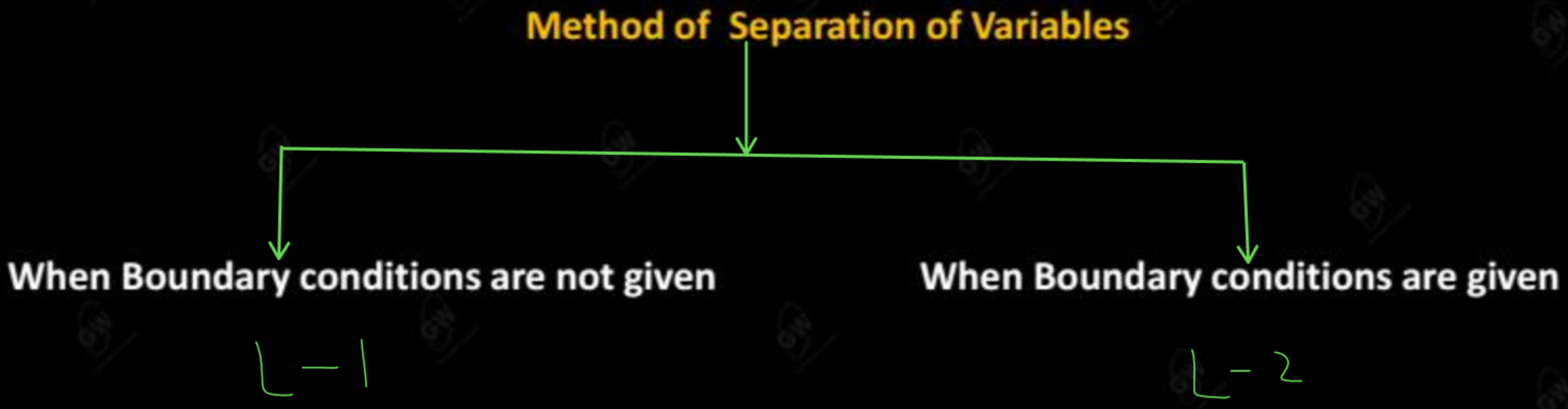
Today's Target

- Method of Separation of Variables (When Boundary conditions are given)
- PYQs
- DPP

are

Method of Separation of Variables

- In this method, we assume the solution of the given PDE as the product of two functions.
- Each function is the function of one variable only.
- So two ordinary differential equations are formed.



When Boundary conditions are given

Q.1:- *Solve the partial differential equation :* $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0; \quad z(0, y) = 2e^{-y}$ *by the method of separation of variables.* **(AKTU-2017)**

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0 \quad \text{—} (1)$$

Let $z(x, y) = XY$ — (2)

$$\frac{\partial z}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial z}{\partial y} = X\frac{dY}{dy} = XY'$$

Put these values in (1)

$$2YX' + 3XY' + 5XY = 0$$

Divide by XY

$$\frac{2YX'}{XY} + \frac{3XY'}{XY} + \frac{5XY}{XY} = 0$$

$$2\frac{X'}{X} + 3\frac{Y'}{Y} + 5 = 0$$

$$\frac{2x'}{x} = -\left(\frac{3y'}{y} + 5\right) = K \text{ (say)}$$

$$\frac{2x'}{x} = K$$

$$2x' = Kx$$

$$2Dx = Kx$$

$$2Dx - Kx = 0$$

$$(2D - K)x = 0$$

Aux. Eqn

$$2D - K = 0$$

$$D = \frac{K}{2}$$

$$CF = c_1 e^{\frac{K}{2}x}$$

$$PI = 0$$

$$\boxed{x = c_1 e^{\frac{K}{2}x}}$$

$$-\left(\frac{3y'}{y} + 5\right) = K$$

$$\frac{3y' + 5y}{y} = -K$$

$$3y' + 5y = -Ky$$

$$3Dy + 5y + Ky = 0$$

$$(3D + 5 + K)y = 0$$

Aux. Eqn

$$3D + 5 + K = 0$$

$$D = -\frac{(K+5)}{3}$$

$$CF = c_2 e^{-\left(\frac{K+5}{3}\right)y}$$

$$PI = 0$$

$Y = CF + PI$

$$Y = C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

Put X and Y in (2)

$$z(x,y) = C_1 e^{\frac{k}{2}x} \times C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

$$z(x,y) = C_1 C_2 e^{\frac{k}{2}x - \left(\frac{k+5}{3}\right)y}$$

(3)

Apply Boundary condition

Put $x = 0$

$$z(0,y) = C_1 C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

$$2e^{-y} = C_1 C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

Where

$$C_1 C_2 = 2$$

$$+1 = +\left(\frac{k+5}{3}\right)$$

$$1 = \frac{k+5}{3}$$

$$k+5 = 3$$

$$k = -2$$

Put $C_1 C_2$ and k in (3)

$$z(x,y) = 2e^{-x-y}$$

$$z(x,y) = 2e^{-(x+y)}$$

Q.2:- Solve by method of separation of variables $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u;\quad u(x,\ 0) = 10e^{-x} - 6e^{-4x}$ **(UPTU-2018)**

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u \quad \text{—— (1)}$$

Let $u = TX$ —— (2)

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt} = XT'$$

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx} = TX'$$

Put these values in (1)

$$XT' = TX' - 2TX$$

Divide by TX

$$\frac{XT'}{TX} = \frac{TX'}{TX} - \frac{2TX}{TX}$$

$$\frac{T'}{T} = \frac{X'}{X} - 2 = p^2 \text{ (say)}$$

$$\frac{T'}{T} = p^2$$

$$T' = p^2 T$$

$$T' - p^2 T = 0$$

$$DT - p^2 T = 0$$

$$(D - p^2)T = 0$$

Aux. Eqn

$$D - p^2 = 0$$

$$D = p^2$$

$$CF = c_1 e^{p^2 t}$$

$$PI = 0$$

$T = CF + PI$

$$T = c_1 e^{p^2 t}$$

$$\frac{X'}{X} - 2 = p^2$$

$$\frac{X' - 2X}{X} = p^2$$

$$X' - 2X = p^2 X$$

$$DX - 2X - p^2 X = 0$$

$$(D - 2 - p^2) X = 0$$

Aux. Eq^n

$$D - 2 - p^2 = 0$$

$$D = p^2 + 2$$

$$CF = c_2 e^{(p^2+2)x}$$

$$PI = 0$$

$$X = c_2 e^{(p^2+2)x}$$

Put T and X in (2)

$$u(t,x) = c_1 e^{p^2 t} \times c_2 e^{(p^2+2)x}$$

$$u(t,x) = c_1 c_2 e^{p^2 t + (p^2+2)x}$$

Most General solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{p^2 t + (p^2+2)x}$$

(3)

Apply Boundary conditions

Put $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} b_n e^{(\beta^2+2)x}$$

$$10e^{-x} - 6e^{-4x} = b_1 e^{(\beta^2+2)x} + b_2 e^{(\beta^2+2)x} + b_3 e^{(\beta^2+2)x} + \cdots$$

Where

$b_1 = 10$ | $b_2 = -6$

$\beta^2 + 2 = -1$ | $\beta^2 + 2 = -4$

$\beta^2 = -3$ | $\beta^2 = -6$

From ③

$$u(x,t) = b_1 e^{p^2 t + (p^2+2)x} + b_2 e^{p^2 t + (p^2+2)x}$$

$$u(x,t) = 10\, e^{-3t-x} - 6\, e^{-6t-4x}$$

$$\boxed{u(x,t) = 10\, e^{-(x+3t)} - 6\, e^{-2(2x+3t)}}$$

Q.3. Solve by method of separation of variables $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0;\ z(x,\ 0) = 0,\ \ z(x,\ \pi) = 0,\ \ z(0,\ y) = 4\ sin\ 3y$ (UPTU-2014)

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{— (1)}$$

Let $z(x,y) = XY$ — (2)

$$\frac{\partial z}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial z}{\partial y} = X\frac{dY}{dy} = XY'$$

$$\frac{\partial^2 z}{\partial y^2} = X\frac{d^2Y}{dy^2} = XY''$$

Put these values in (1)

$$YX' + XY'' = 0$$

Divide by XY

$$\frac{YX'}{XY} + \frac{XY''}{XY} = 0$$

$$\frac{X'}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X'}{X} = -\frac{Y''}{Y} = p^2 \text{ (say)}$$

$$\frac{X'}{X} = \beta^2$$

$$X' = \beta^2 X$$

$$X' - \beta^2 X = 0$$

$$DX - \beta^2 X = 0$$

$$(D - \beta^2)X = 0$$

Aux. Eqn

$$D - \beta^2 = 0$$

$$D = \beta^2$$

$$CF = c_1 e^{\beta^2 x}$$

$$PI = 0$$

$$\boxed{X = c_1 e^{\beta^2 x}}$$

$$\frac{-Y''}{Y} = \beta^2$$

$$-Y'' = \beta^2 Y$$

$$Y'' + \beta^2 Y = 0$$

$$D^2 Y + \beta^2 Y = 0$$

$$(D^2 + \beta^2)Y = 0$$

Aux. Eqn

$$D^2 + \beta^2 = 0$$

$$D^2 = -\beta^2$$

$$D = \sqrt{-\beta^2}$$

$$D = \pm i\beta$$

$$D = 0 \pm i\beta$$

$$CF = e^{\alpha y}(c_2 \cos\beta y + c_3 \sin\beta y)$$

$$CF = c_2 \cos\beta y + c_3 \sin\beta y$$

$$PI = 0$$

$$\boxed{Y = c_2 \cos\beta y + c_3 \sin\beta y}$$

Put X and Y in (2)

$$z(x,y) = c_1 e^{p^2 x} \times (c_2 \cos py + c_3 \sin py) \quad \text{— (3)}$$

Given Boundary conditions

$z(x,0) = 0$ — (i)

$z(x,\pi) = 0$ — (ii)

$z(0,y) = 4 \sin 3y$ — (iii)

Apply Boundary conditions (i)

Put $y = 0$

$z(x,0) = c_1 e^{p^2 x} c_2$

$0 = c_1 c_2 e^{p^2 x}$

$c_1 c_2 e^{p^2 x} = 0$

$c_1 \neq 0$

$e^{p^2 x} \neq 0$

$\Rightarrow c_2 = 0$

Put $c_2 = 0$ in (3)

$$z(x,y) = c_1 e^{p^2 x} \times c_3 \sin py$$

$$z(x,y) = c_1 c_3 e^{p^2 x} \sin py \quad \text{— (4)}$$

Apply Boundary condition (ii)

Put $y = \pi$

$$z(x,\pi) = c_1 c_3 e^{p^2 x} \sin p\pi$$

$$c_1 c_3 e^{p^2 x} \sin p\pi = 0$$

$$c_1 \neq 0$$

$$c_3 \neq 0$$

$$e^{p^2 x} \neq 0$$

$$\Rightarrow \sin p\pi = 0$$

$$p\pi = n\pi$$

$$p = n$$

Put $p = n$ in (4)

$$\boxed{z(x,y) = c_1 c_3 e^{n^2 x} \sin ny} \quad \text{— (5)}$$

Apply (iii) Boundary condition

Put $x = 0$

$$z(0,y) = c_1 c_3 e^{0} \sin ny$$

$$4 \sin 3y = c_1 c_3 \sin ny$$

Where

$$c_1 c_3 = 4$$

$$n = 3$$

$$z(x,y) = 4e^{9x} \sin 3y$$

➢ Topic : Method of Separation of Variables (When Boundary conditions are given)

Q .1 *Solve by the method of separation of variables* $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0;\ u(x,\ 0) = 4e^{-x}$ *(AKTU-2017)*

Ans. $u(x,\ y) = 4e^{-x+3y/2}$

Q .2 *Solve by the method of separation of variables* $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u;\ u(0,\ y) = 4e^{-y} - e^{-5y}$ *(AKTU-2022)*

Ans. $u(x,\ y) = 4e^{+x-y} - e^{2x-5y}$

Q .3 *Use the method of separation of variables to solve the equation* $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$ *given that* $v = 0$ *when* $t \to \infty$ *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$. *(GBTU-2013)*

Ans. $u(x,\ y) = (c_1 \cos px + c_2 \sin px)c_3 e^{-(p^2+2)y}$

Q .4 *Solve the P.D.E. by separation of variables method* $u_{xx} = u_y + 2u,\ u(0,\ y) = 0, \frac{\partial}{\partial x} u(0,\ y) = 1 + e^{-3y}$. *(GBTU-2013)*

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