ENGINEERING MATHEMATICS

UNIT-4 : *Statistical Techniques – II*

Lec-06

Today's Target

- $Poisson\ \ Probability\ Distribution(Part - I)$
- PYQs
- DPPs

Poisson Probability Distribution

- Poisson Distribution is a limiting case of binomial distribution

$$P(X = r) = \frac{\lambda^r e^{-\lambda}}{r!} \qquad (r = 0, 1, 2, 3, \dots)$$

- Recurrence formula for the Poisson Distribution

$$P(r+1) = \frac{\lambda}{(r+1)} P(r) \qquad (r = 0, 1, 2, 3, \dots.)$$

- Mean and Variance of the Poisson Distribution

$$Mean = \lambda$$

$$Variance = \lambda$$

1. Show that Poisson Distribution is a limiting case of binomial distribution (A.K.T.U. 2021)

GW GATEWAY CLASSES

In Binomial Distribution

n = number of trials

P = Prob. of success

- If n is very large and p is very small then Binomial distribution become complicated
- If we assume that $n \to \infty$ and $p \to 0$ such that $\boxed{np = \lambda}$ is finite

Then we get poisson approximation to binomial distribution.

$$\boxed{P(x=r) = \frac{e^{-\lambda}\,\lambda^{r}}{r!}}$$

Proof :

$$P(x=r) = {}^{n}C_{r}\, p^{r}\, q^{n-r}$$

$$P(X=r) = {}^{n}C_{r}\, p^{r}(1-p)^{n-r} \quad \{p+q=1\}$$

$$P(X=r) = \frac{n!}{(n-r)!\,r!}\, p^{r} \times (1-p)^{n}(1-p)^{-r}$$

$$P(X=r) = \frac{n!}{(n-r)!\,r!} \times \left(\frac{\lambda}{n}\right)^{r} \times \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{r}}$$

$$P(X=r) = \frac{\lambda^{r} \times n(n-1)(n-2)\cdots\cdots(n-r+1)\,(n-r)!\,\left(1-\frac{\lambda}{n}\right)^{n}}{r!\times (n-r)!\times n^{r}\times\left(1-\frac{\lambda}{n}\right)^{r}}$$

$$P(X=r) = \frac{\lambda^{r}}{r!} \times \frac{n}{n} \times \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-r+1}{n}\right)\frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{r}}$$

$$P(X=r) = \frac{\lambda^{r}}{r!} \times 1 \times \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r}{n}+\frac{1}{n}\right)\frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{r}}$$

$$P(X=r) = \lim_{n\to\infty} \frac{\lambda^r}{r!}\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{r}{n}+\frac{1}{n}\right)}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}(1-0)(1-0)\cdots\cdots(1-0+0)\frac{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!} \times 1 \times 1 \cdots\cdots \times 1 \times \frac{e^{-\lambda}}{1}$$

$$= \frac{\lambda^r}{r!} \times e^{-\lambda}$$

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}$$

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^r = 1$$

$$P(X=r) = \frac{\lambda^r e^{-\lambda}}{r!}$$

Poisson Distribution

Poisson Distribution

$$P(X = r) = \frac{\lambda^r e^{-\lambda}}{r!} \qquad (r = 0, 1, 2, 3, \dots)$$

- λ *is called the parameter of the distribution.*

λ = np (Finite)

- *The sum of the Probability* $P(r)$ *for* $r = 0, 1, 2, 3, \dots \dots$ *is* 1

$$\sum P(x) = 1$$

P(X=0) + P(X=1) + P(X=2) + P(X=3) . . . . . = 1

2. Recurrence formula for the Poisson Distribution

By poisson Distribution

$$P(r) = \frac{\lambda^r e^{-\lambda}}{r!} \quad \text{——} \ (1)$$

Replace r by $r+1$

$$P(r+1) = \frac{\lambda^{r+1} e^{-\lambda}}{(r+1)!}$$

$$P(r+1) = \frac{\lambda^r \times \lambda^1 \times e^{-\lambda}}{(r+1) \times r!} \quad \text{——} \ (2)$$

Divide (2) by (1)

$$\frac{P(r+1)}{P(r)} = \frac{\lambda^r \times \lambda \times e^{-\lambda}}{(r+1) \times r!} \times \frac{r!}{\lambda^r \times e^{-\lambda}}$$

$$\frac{P(r+1)}{P(r)} = \frac{\lambda}{r+1}$$

$$\boxed{P(r+1) = \frac{\lambda}{r+1} P(r)}$$

3. Mean and Variance of the Poisson Distribution

By Poisson Distribution

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Mean

$$\mu = E(x) = \sum_{x=0}^{\infty} x\,p(x)$$

$$\mu = \sum_{x=0}^{\infty} \frac{x\,e^{-\lambda}\lambda^{x}}{x!}$$

$$\mu = e^{-\lambda} \sum_{x=1}^{\infty} \frac{x\,\lambda^{x}}{x(x-1)!}$$

$$\mu = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x}}{(x-1)!}$$

$$\mu = e^{-\lambda}\left(\frac{\lambda}{1} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \frac{\lambda^4}{3!} + \cdots\right)$$

$$\mu = e^{-\lambda}\lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)$$

$$\text{Mean} = e^{-\lambda} \times \lambda \times e^{\lambda} = \lambda \times e^{0}$$

$$\boxed{\text{Mean} = \lambda}$$

Variance

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$\sigma^2 = \sum_{x=0}^{\infty} x^2 P(x) - (\lambda)^2$$

$$\sigma^2 = \sum_{x=0}^{\infty} \frac{x^2 e^{-\lambda} \lambda^x}{x!} - \lambda^2$$

$$\sigma^2 = e^{-\lambda} \sum_{x=1}^{\infty} \frac{x^2 \lambda^x}{x!} - \lambda^2$$

$$\sigma^2 = e^{-\lambda}\left(\frac{1^2 \times \lambda}{1!} + \frac{2^2 \times \lambda^2}{2!} + \frac{3^2 \times \lambda^3}{3!} + \dots\right) - \lambda^2$$

$$\sigma^2 = e^{-\lambda} \times \lambda\left(1 + \frac{2\lambda}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \dots\right) - \lambda^2$$

$$\sigma^2 = \lambda e^{-\lambda}\left[1 + \frac{\lambda+\lambda}{1!} + \frac{\lambda^2+2\lambda^2}{2!} + \frac{\lambda^3+3\lambda^3}{3!} + \dots\right) - \lambda^2$$

$$\sigma^2 = \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{2\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{3\lambda^3}{3!} + \dots\right) - \lambda^2$$

$$\sigma^2 = \lambda e^{-\lambda}\left[\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots\right) + \left(\frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} + \dots\right)\right) - \lambda^2$$

$$\sigma^2 = \lambda e^{-\lambda}\left[\left(1 + \frac{\lambda}{1!_0} + \frac{\lambda^2}{2!_0} + \frac{\lambda^3}{3!_0} + \cdots\right) + \lambda\left(1 + \frac{2\lambda}{2!_0} + \frac{3\lambda^2}{3!_0} + \frac{4\lambda^3}{4!_0} + \cdots\right)\right] - \lambda^2$$

$$\sigma^2 = \lambda e^{-\lambda}\left[e^{\lambda} + \lambda\left(1 + \frac{\lambda}{1!_0} + \frac{\lambda^2}{2!_0} + \frac{\lambda^3}{3!_0} + \cdots\right)\right] - \lambda^2$$

$$\sigma^2 = \lambda e^{-\lambda}\left[e^{\lambda} + \lambda e^{\lambda}\right] - \lambda^2$$

$$\sigma^2 = \lambda e^{-\lambda} e^{\lambda} + \lambda^2 e^{-\lambda} e^{\lambda} - \lambda^2$$

$$\sigma^2 = \lambda + \cancel{\lambda^2} - \cancel{\lambda^2}$$

$$\sigma^2 = \lambda$$

$$\text{variance} = \lambda$$

Q.1 *If a random variable X follow a poisson distribution such that* $P(X=2) = 9P(X=4) + 90P(X=6)$. *Find mean and variance of X.* **(A.K.T.U. 2023)**

For poisson Distribution

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=2) = 9P(X=4) + 90P(X=6)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9 \times \frac{e^{-\lambda}\lambda^4}{4!} + \frac{90 \times e^{-\lambda}\lambda^6}{6!}$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = e^{-\lambda}\left(\frac{9\lambda^4}{4!} + \frac{90\lambda^6}{6!}\right)$$

$$\frac{\lambda^2}{2} = \lambda^2\left(\frac{3\lambda^2}{4\times3\times2\times1} + \frac{30\lambda^4}{6\times5\times4\times3\times2\times1}\right)$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$\frac{1}{2} = \frac{3\lambda^2 + \lambda^4}{8}$$

$$4 = 3\lambda^2 + \lambda^4$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

Put $\lambda^2 = x$

$x^2 + 3x - 4 = 0$

$x^2 + 4x - x - 4 = 0$

$x(x+4) - 1(x+4) = 0$

$(x-1)(x+4) = 0$

$x = 1$ | $x = -4$

$\lambda^2 = 1$ | $\lambda^2 = -4$

$\lambda = \pm 1$ | (Reject)

$\lambda = 1$

Hence

$\lambda = 1$

Mean $= 1$

Variance $= 1$

Q. 2 *If the variance of the Poisson distribution is 2, find the probabilities for* $r = 1, 2, 3, 4$ *from the recurrence relation of the Poisson distribution. Also, find* $P(r \geq 4)$.

(M.T.U. 2013)

Given

$\lambda = 2$

By poisson Distribution

$$P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$$P(0) = \frac{e^{-2} \times \lambda^0}{0!}$$

$$P(0) = e^{-2}$$

$$P(0) = 0.1353$$

By Recurrence Relation

$$P(r+1) = \frac{\lambda}{r+1} P(r)$$

$$P(0+1) = \frac{2}{0+1} P(0)$$

$$P(1) = 2\, P(0)$$

$$\boxed{P(1) = 0.2706}$$

$$P(1+1) = \frac{2}{1+1} P(1)$$

$$P(2) = P(1)$$

$$\boxed{P(2) = 0.2706}$$

$$P(2+1) = \frac{2}{2+1} P(2)$$

$$P(3) = \frac{2}{3} \times 0.2706$$

$$\boxed{P(3) = 0.1804}$$

$$P(3+1) = \frac{2}{3+1} P(3)$$

$$P(4) = \frac{1}{2} \times 0.1804$$

$$\boxed{P(4) = 0.0902}$$

$$P(x \geq 4) = 1 - P(x < 4)$$

$$P(x \geq 4) = 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$P(x \geq 4) = 1 - 0.8569$$

$$\boxed{P(x \geq 4) = 0.1431}$$

Q.3 *Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well – shuffled cards at least once in 104 consecutive trials.* **(U.P.T.U. 2015)**

$P(\text{ace of spades}) = \frac{1}{52}$

$p = \frac{1}{52}$

$n = 104$

$\lambda = np$

$\lambda = 104 \times \frac{1}{52}$

$\lambda = 2$

By Poisson Distribution

$$P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$$P(r \geq 1) = 1 - P(x = 0)$$

$$= 1 - \frac{e^{-2} \times \lambda^0}{0!} = 1 - e^{-2}$$

$$P(r \geq 1) = 0.8647$$

DPP- 06

Topic :$Poisson\ \ Probability\ Distribution\ (Part-I)$

$Q.1$ $In\ a\ poisson\ distribution, P(r) for\ r = 0\ is\ 10\%, find\ the\ mean.$

$Q.2$ $If\ X\ poisson\ variate\ such\ that\ P(X = 1) = P(X = 2). Find\ \ (i)\ Mean\ (ii) P(X = 4)$

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Thank You