Engineering Mathematics-IV

Module -II: Applications of Partial Differential Equations and Fourier Transform

Chapter-1 :- Applications of Partial Differential Equations

- Method of separation of variables
- Solution of one dimensional heat equation, wave equation
- Two dimensional heat equation (only Laplace Equation) and their application

Chapter-2 :- Fourier Transform

- Complex Fourier transform, Fourier sine transform, Fourier cosine transform, Inverse transform, convolution theorem
- Application of Fourier Transformto solve partial differential equation.

Gateway Classes: 7455 961284

Engineering Mathematics

UNIT : Applications of Partial Differential Equations and Fourier Transform

Today's Target

- Method of separation of variables(When boundary conditions are not given)
- PYQs
- DPP

Gateway Classes : 7455 9612 84

Calculation of Complementary Function (C.F)

CASE-1 : When A.E has Real and distinct roots

Nature of roots	C.F
(i) One real root m_1	$CF = c_1 e^{m_1 x}$
(ii) Two real and distinct roots m_1, m_2	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
(iii) Three real and distinct roots m_1, m_2, m_3	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

CASE-2 : When A.E has Real and equal roots

Nature of roots	C.F
(i) Two real and equal roots m, m	$CF = (c_1 + c_2 x)e^{mx}$
(ii) Three real and equal roots m, m, m	$CF = (c_1 + c_2 x + c_3 x^2)e^{mx}$

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CASE-3 : When A.E has *complex* roots

Nature of roots	C.F
(i) One pair of complex roots $\alpha \pm i\beta$	$CF = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
(ii) Two pair of complex and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$	$CF = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x]$

CASE-4 : When A.E has *Irrational* roots

Nature of roots	C.F
(i) One pair of irrational roots $\alpha \pm \sqrt{\beta}$	$CF = c_1 e^{(\alpha+\beta)x} + c_2 e^{(\alpha-\beta)x}$
(ii) Two pair of irrational and equal roots $\alpha \pm \sqrt{\beta}, \quad \alpha \pm \sqrt{\beta}$	$CF = (c_1 + c_2 x)e^{(\alpha+\beta)x} + (c_3 + c_4 x)e^{(\alpha-\beta)x}$

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Method of separation of variables

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- *In this method, we assume the solution of the given PDE as the product of the two functions.*
- *Each function is the function of one variable only.*
- *So two ordinary differential equations are formed.*

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Method of separation of variables

- *When boundary conditions are not given* — L-1
- *When boundary conditions are given* — L-2

When boundary conditions are not given

Q.1. Use the method of separation of variables to solve the equation $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$. (GBTU-2012)

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \text{—— (1)}$$

Let $u = X(x)\,Y(y)$

$$u = XY \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = YX''$$

$$\frac{\partial u}{\partial y} = X\frac{dY}{dy} = XY'$$

Put these values in (1)

$$YX'' - 2YX' + XY' = 0 \quad \text{— (3)}$$

Divide by XY

Gateway Classes 7455 961234

$$\frac{XX''}{XY} - \frac{2XX'}{XY} + \frac{XY'}{XY} = \frac{0}{XY}$$

$$\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0$$

$$\frac{X''}{X} - 2\frac{X'}{X} = -\frac{Y'}{Y}$$

$$\frac{X''-2X'}{X} = -\frac{Y'}{Y} = -p^2 \text{ (say)} \quad \text{--- (1)}$$

$$\frac{X''-2X'}{X} = -p^2$$

$$X''-2X' = -p^2X$$

$$X''-2X'+p^2X = 0$$

$$D^2X - 2DX + p^2X = 0$$

$$(D^2-2D+p^2)X = 0$$

Aux. Eqn

$$D^2-2D+p^2 = 0$$

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$$D^2 - 2D + p^2 = 0$$

$$D = \frac{2 \pm \sqrt{(-2)^2 - 4 \times p^2}}{2 \times 1}$$

$$D = \frac{2 \pm \sqrt{4 - 4p^2}}{2}$$

$$D = \frac{2 \pm 2\sqrt{1-p^2}}{2}$$

$$D = 1 \pm \sqrt{1-p^2}$$

$$CF = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}$$

$$PI = 0$$

$$X = CF + PI$$

$$\boxed{X = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}}$$

From (4)

$$\frac{-Y'}{Y} = -p^2$$

$$+Y' = +p^2 Y$$

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$$Y' = p^2 Y$$

$$Y' - p^2 Y = 0$$

$$DY - p^2 Y = 0$$

$$(D - p^2) Y = 0$$

Aux. Eqn

$$D - p^2 = 0$$

$$D = p^2$$

$$CF = C_3 e^{p^2 y}$$

$$PI = 0$$

$$Y = CF + PI$$

$$\boxed{Y = C_3 e^{p^2 y}}$$

Put X and Y in (2)

$$\boxed{u = \left[C_1 e^{(1+\sqrt{1-p^2})x} + C_2 e^{(1-\sqrt{1-p^2})x} \right] C_3 e^{p^2 y}}$$

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Q.2. *Use the method of separation of variables to solve the equation* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$.

(UPTU-2014)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad \text{—— (1)}$$

Let $u = X(x)\,Y(y)$

$$u = XY \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = YX''$$

$$\frac{\partial u}{\partial y} = X\frac{dY}{dy} = XY'$$

Put these values in (1)

$$YX'' = XY' + 2XY$$

Divide by XY

$$\frac{YX''}{XY} = \frac{XY'}{XY} + \frac{2XY}{XY}$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2$$

$$\boxed{\frac{X''}{X} = \frac{Y'}{Y} + 2 = -p^2 \text{ (say)}} \quad \text{—— (3)}$$

$$\frac{X''}{X} = -p^2$$

$$X'' = -p^2 X$$

$$X'' + p^2 X = 0$$

$$D^2X + p^2X = 0$$

Gateway Classes · 7455 962 84

$(D^2+b^2)X=0$

Aux. Eqn

$D^2+b^2=0$

$D=\pm ib$

$D=0\pm ib$

$CF=e^{0x}(c_1\cos bx+c_2\sin bx)$

$CF=c_1\cos bx+c_2\sin bx$

$PI=0$

$X=CF+PI$

$$\boxed{X=c_1\cos bx+c_2\sin bx}$$

From (3)

$$\frac{Y'}{Y}+2=-b^2$$

$$\frac{Y'+2Y}{Y}=-b^2$$

$$Y'+2Y=-b^2Y$$

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$$Y' + 2Y + p^2 Y = 0$$

$$DY + 2Y + p^2 Y = 0$$

$$(D + 2 + p^2) Y = 0$$

Aux. Eqn

$$D + 2 + p^2 = 0$$

$$D = -(p^2 + 2)$$

$$CF = C_3 e^{-(p^2+2)y}$$

$$PI = 0$$

$$Y = CF + PI$$

$$\boxed{Y = C_3 e^{-(p^2+2)y}}$$

Put X and Y in (2)

$$\boxed{u = (C_1 \cos px + C_2 \sin px) C_3 e^{-(p^2+2)y}}$$

Gateway Classes : 7455 961284

Topic : Method of separation of variables(When boundary conditions are not given)

Q .1. *Use the method of separation of variables to solve the equation* $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ *UPTU-2012*

Ans. $u(x,\ y) = c_1 c_2\ e^{k(x+y)}$

Q.2. *Use the method of separation of variables to solve the equation* $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$ *UPTU-2007*

Ans. $u(x,\ y) = c_1\ e^{-p^2 y}\ (c_2\ cos\ px + c_3\ sin\ px)$

Gateway Classes : 7455 961 284

Engineering Mathematics

UNIT : Application of Partial Differential Equation and Fourier Transform

Today's Target

- Method of Separation of Variables (When Boundary conditions are given)
- PYQs
- DPP

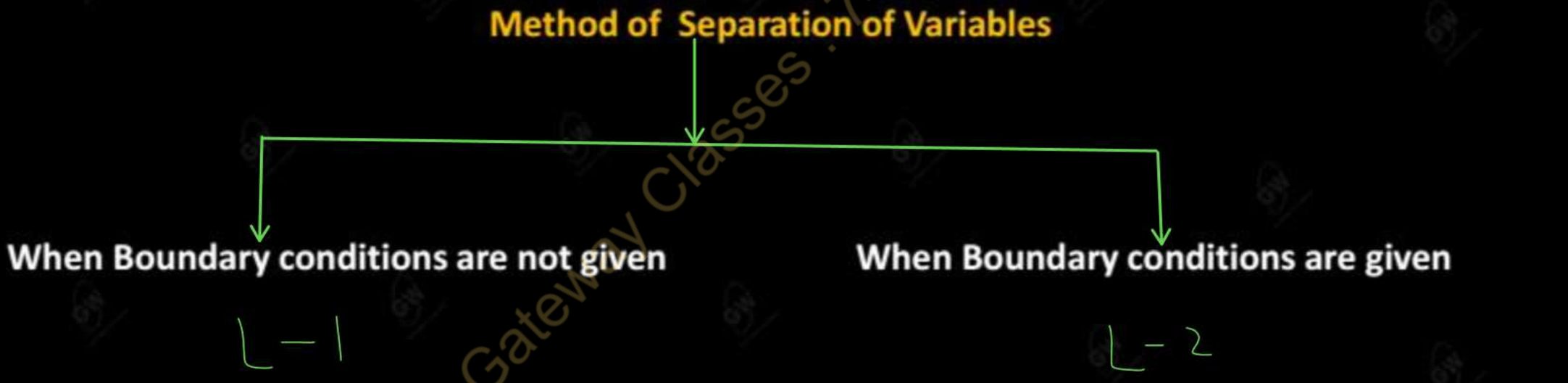
Gateway Classes : 7455 9842 84

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Method of Separation of Variables

- In this method, we assume the solution of the given PDE as the product of two functions.
- Each function is the function of one variable only.
- So two ordinary differential equations are formed.

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When Boundary conditions are given

Q.1:- *Solve the partial differential equation :* $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0; \quad z(0, y) = 2e^{-y}$ *by the method of separation of variables.* **(AKTU-2017)**

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0 \quad \text{—} ①$$

Let $z(x, y) = XY$ — ②

$$\frac{\partial z}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial z}{\partial y} = X\frac{dY}{dy} = XY'$$

Put these values in ①

$$2YX' + 3XY' + 5XY = 0$$

Divide by XY

$$\frac{2YX'}{XY} + \frac{3XY'}{XY} + \frac{5XY}{XY} = 0$$

$$2\frac{X'}{X} + 3\frac{Y'}{Y} + 5 = 0$$

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$$\frac{2x'}{x} = -\left(\frac{3y'}{y} + 5\right) = k \text{ (say)}$$

$$\frac{2x'}{x} = k$$

$$2x' = kx$$

$$2Dx = kx$$

$$2Dx - kx = 0$$

$$(2D - k)x = 0$$

Aux. Eqn

$$2D - k = 0$$

$$D = \frac{k}{2}$$

$$CF = c_1 e^{\frac{k}{2}x}$$

$$PI = 0$$

$$\boxed{x = c_1 e^{\frac{k}{2}x}}$$

$$-\left(\frac{3y'}{y} + 5\right) = k$$

$$\frac{3y' + 5y}{y} = -k$$

$$3y' + 5y = -ky$$

$$3Dy + 5y + ky = 0$$

$$(3D + 5 + k)y = 0$$

Aux. Eqn

$$3D + 5 + k = 0$$

$$D = -\frac{(k+5)}{3}$$

$$CF = c_2 e^{-\left(\frac{k+5}{3}\right)y}$$

$$PI = 0$$

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Gateway Classes 7455 96 12 84

$Y = CF + PI$

$$Y = C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

Put X and Y in (2)

$$z(x,y) = C_1 e^{\frac{k}{2}x} \times C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

$$z(x,y) = C_1 C_2 e^{\frac{k}{2}x - \left(\frac{k+5}{3}\right)y}$$

└ (3)

Apply Boundary condition

Put $x = 0$

$$z(0,y) = C_1 C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

$$2e^{-y} = C_1 C_2 e^{-\left(\frac{k+5}{3}\right)y}$$

Where

$$C_1 C_2 = 2$$

$$+1 = +\left(\frac{k+5}{3}\right)$$

$$1 = \frac{k+5}{3}$$

$$k+5 = 3$$

$$k = -2$$

Put $C_1 C_2$ and k in (3)

$$z(x,y) = 2e^{-x-y}$$

$$z(x,y) = 2e^{-(x+y)}$$

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Q.2:- Solve by method of separation of variables $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$; $u(x, 0) = 10e^{-x} - 6e^{-4x}$ *(UPTU-2018)*

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u \quad \text{—— (1)}$$

Let $u = TX$ —— (2)

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt} = XT'$$

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx} = TX'$$

Put these values in (1)

$$XT' = TX' - 2TX$$

Divide by TX

$$\frac{XT'}{TX} = \frac{TX'}{TX} - \frac{2TX}{TX}$$

$$\frac{T'}{T} = \frac{X'}{X} - 2 = p^2 \text{ (say)}$$

$$\frac{T'}{T} = p^2$$

$$T' = p^2 T$$

$$T' - p^2 T = 0$$

$$DT - p^2 T = 0$$

$$(D - p^2)T = 0$$

Aux. Eqn

$$D - p^2 = 0$$

$$D = p^2$$

$$CF = c_1 e^{p^2 t}$$

$$PI = 0$$

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$T = CF + PI$

$$T = c_1 e^{p^2 t}$$

$$\frac{X'}{X} - 2 = p^2$$

$$\frac{X' - 2X}{X} = p^2$$

$$X' - 2X = p^2 X$$

$$DX - 2X - p^2 X = 0$$

$$(D - 2 - p^2) X = 0$$

Aux. Eqn

$$D - 2 - p^2 = 0$$

$$D = p^2 + 2$$

$$CF = c_2 e^{(p^2+2)x}$$

$$PI = 0$$

$$X = c_2 e^{(p^2+2)x}$$

Put T and X in (2)

$$u(t,x) = c_1 e^{p^2 t} \times c_2 e^{(p^2+2)x}$$

$$u(t,x) = c_1 c_2 e^{p^2 t + (p^2+2)x}$$

Most General solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{p^2 t + (p^2+2)x} \quad \text{— (3)}$$

Apply Boundary conditions

Put $t = 0$

GW GATEWAY CLASSES
Gateway Classes : 7455961234

$$u(x,0) = \sum_{n=1}^{\infty} b_n e^{(\beta^2+2)x}$$

$$10e^{-x} - 6e^{-4x} = b_1 e^{(\beta^2+2)x} + b_2 e^{(\beta^2+2)x} + b_3 e^{(\beta^2+2)x} + \cdots$$

Where

$b_1 = 10$ | $b_2 = -6$

$\beta^2 + 2 = -1$ | $\beta^2 + 2 = -4$

$\beta^2 = -3$ | $\beta^2 = -6$

Gateway Classes : 7455 961234

From ③

$$u(x,t) = b_1 e^{p^2 t + (p^2+2)x} + b_2 e^{p^2 t + (p^2+2)x}$$

$$u(x,t) = 10\, e^{-3t-x} - 6\, e^{-6t-4x}$$

$$\boxed{u(x,t) = 10\, e^{-(x+3t)} - 6\, e^{-2(2x+3t)}}$$

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Q.3. Solve by method of separation of variables $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0;\ z(x,\ 0) = 0,\ \ z(x,\ \pi) = 0,\ \ z(0,\ y) = 4\ sin\ 3y$ (UPTU-2014)

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{— (1)}$$

Let $z(x,y) = XY$ — (2)

$$\frac{\partial z}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial z}{\partial y} = X\frac{dY}{dy} = XY'$$

$$\frac{\partial^2 z}{\partial y^2} = X\frac{d^2Y}{dy^2} = XY''$$

Put these values in (1)

$$YX' + XY'' = 0$$

Divide by XY

$$\frac{YX'}{XY} + \frac{XY''}{XY} = 0$$

$$\frac{X'}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X'}{X} = -\frac{Y''}{Y} = p^2 \text{ (say)}$$

Gateway Classes: 7455 9612 84

$$\frac{X'}{X} = \beta^2$$

$$X' = \beta^2 X$$

$$X' - \beta^2 X = 0$$

$$DX - \beta^2 X = 0$$

$$(D - \beta^2)X = 0$$

Aux. Eqn

$$D - \beta^2 = 0$$

$$D = \beta^2$$

$$CF = c_1 e^{\beta^2 x}$$

$$PI = 0$$

$$\boxed{X = c_1 e^{\beta^2 x}}$$

$$\frac{-Y''}{Y} = \beta^2$$

$$-Y'' = \beta^2 Y$$

$$Y'' + \beta^2 Y = 0$$

$$D^2 Y + \beta^2 Y = 0$$

$$(D^2 + \beta^2)Y = 0$$

Aux. Eqn

$$D^2 + \beta^2 = 0$$

$$D^2 = -\beta^2$$

$$D = \sqrt{-\beta^2}$$

$$D = \pm i\beta$$

$$D = 0 \pm i\beta$$

$$CF = e^{\alpha y}(c_2 \cos\beta y + c_3 \sin\beta y)$$

$$CF = c_2 \cos\beta y + c_3 \sin\beta y$$

$$PI = 0$$

$$\boxed{Y = c_2 \cos\beta y + c_3 \sin\beta y}$$

Put X and Y in (2)

GW GATEWAY CLASSES
Gateway Classes: 7455 9612 84

$$z(x,y) = c_1 e^{p^2 x} \times (c_2 \cos py + c_3 \sin py) \quad \text{— (3)}$$

Given Boundary conditions

$z(x,0) = 0$ — (i)

$z(x,\pi) = 0$ — (ii)

$z(0,y) = 4\sin 3y$ — (iii)

Apply Boundary conditions (i)

Put $y = 0$

$z(x,0) = c_1 e^{p^2 x} c_2$

$0 = c_1 c_2 e^{p^2 x}$

$c_1 c_2 e^{p^2 x} = 0$

$c_1 \neq 0$

$e^{p^2 x} \neq 0$

$\Rightarrow c_2 = 0$

Put $c_2 = 0$ in (3)

Gateway Classes : 7455 961284

$$z(x,y) = c_1 e^{p^2 x} \times c_3 \sin py$$

$$z(x,y) = c_1 c_3 e^{p^2 x} \sin py \quad \text{— (4)}$$

Apply Boundary condition (ii)

Put $y = \pi$

$$z(x,\pi) = c_1 c_3 e^{p^2 x} \sin p\pi$$

$$c_1 c_3 e^{p^2 x} \sin p\pi = 0$$

$c_1 \neq 0$

$c_3 \neq 0$

$e^{p^2 x} \neq 0$

$\Rightarrow \sin p\pi = 0$

$p\pi = n\pi$

$p = n$

Put $p = n$ in (4)

$$\boxed{z(x,y) = c_1 c_3 e^{n^2 x} \sin ny} \quad \text{— (5)}$$

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Apply (iii) Boundary condition

Put $x = 0$

$$z(0,y) = c_1 c_3 e^0 \sin ny$$

$$4\sin 3y = c_1 c_3 \sin ny$$

Where

$$c_1 c_3 = 4$$

$$n = 3$$

$$z(x,y) = 4e^{9x} \sin 3y$$

Gateway Classes : 7455 9612 84

➢ Topic : Method of Separation of Variables (When Boundary conditions are given)

Q .1 *Solve by the method of separation of variables* $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0;\ u(x,\ 0) = 4e^{-x}$ *(AKTU-2017)*

Ans. $u(x,\ y) = 4e^{-x+3y/2}$

Q .2 *Solve by the method of separation of variables* $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u;\ u(0,\ y) = 4e^{-y} - e^{-5y}$ *(AKTU-2022)*

Ans. $u(x,\ y) = 4e^{+x-y} - e^{2x-5y}$

Q .3 *Use the method of separation of variables to solve the equation* $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$ *given that* $v = 0$ *when* $t \to \infty$ *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$. *(GBTU-2013)*

Ans. $u(x,\ y) = (c_1 \cos px + c_2 \sin px)c_3 e^{-(p^2+2)y}$

Q .4 *Solve the P.D.E. by separation of variables method* $u_{xx} = u_y + 2u,\ u(0,\ y) = 0, \frac{\partial}{\partial x} u(0,\ y) = 1 + e^{-3y}$. *(GBTU-2013)*

Gateway Classes : 7455 961 984

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (PART-1
- PYQ
- DPP

Gateway Classes : 7455 9612 84

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

One Dimensional wave equation

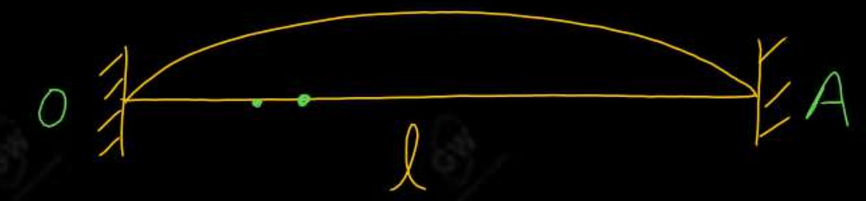
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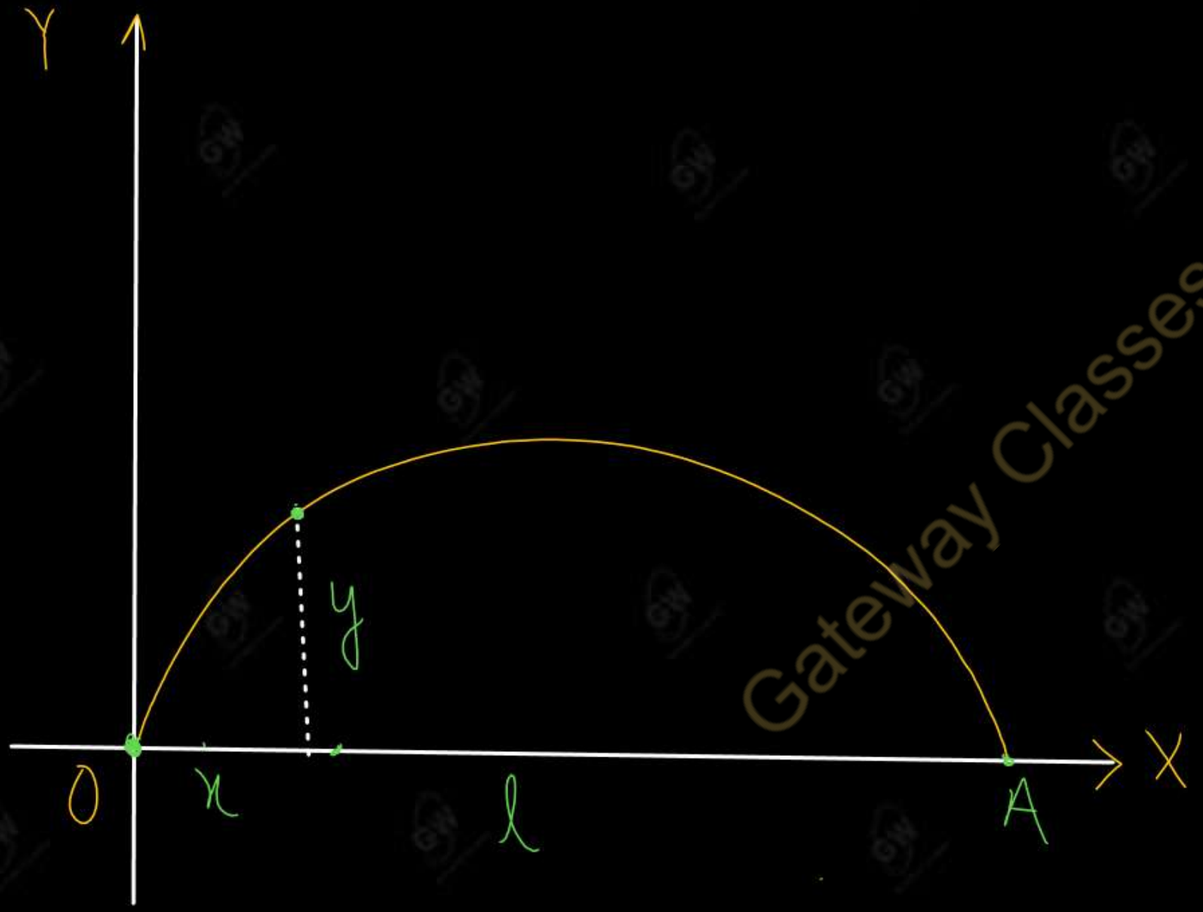
Equation of vibrating string

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Gateway Classes : 7455 9612 84

Vibrations of a stretched string





Boundary conditions

(i) At $x = 0$, $y = 0$

(ii) At $x = l$, $y = 0$

Initial condition

At $t = 0$, $y = f(x)$

At $t = 0$, $\frac{dy}{dt} = 0$

Gateway Classes : 7455961284

Q.1. Solve one dimensional wave equation by using separation of variables.

Method of

one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

Let $y = X(x)T(t)$

$$y = XT \quad \text{— (2)}$$

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2} = XT''$$

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{\partial x^2} = TX''$$

Put these values in (1)

$$XT'' = c^2 TX''$$

$$XT'' = c^2 TX''$$

Divide by XT

$$\frac{XT''}{XT} = c^2 \frac{TX''}{XT}$$

Gateway Classes : 7455 962 84

$$\frac{T''}{T} = c^2 \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T}$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{— (3)}$$

$$\frac{1}{c^2} \frac{T''}{T} = K$$

$$T'' = c^2 K T$$

$$T'' - c^2 K T = 0$$

$$D^2T - c^2 KT = 0$$

$$(D^2 - c^2 K)T = 0 \quad \text{— (4)}$$

CASE-1 : $K = 0$

From (3)

$$D^2X = 0$$

Aux Eqn

$$D^2 = 0$$

$$D = 0, 0$$

$$X = C_1 + C_2 x$$

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From (4)

$$D^2T = 0$$

Aux. Eqn

$$D^2 = 0$$

$$D = 0, 0$$

$$T = C_3 + C_4 t$$

Put X and T in (2)

$$y = (C_1 + C_2 x)(C_3 + C_4 t) \quad \text{—— (5)}$$

CASE-2: $K = p^2$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D = \pm p$$

$$X = C_1 e^{px} + C_2 e^{-px}$$

From (4)

$$(D^2 - c^2 p^2)T = 0$$

$$D^2 - c^2 p^2 = 0$$

$$D = \pm cp$$

$$T = C_3 e^{cpt} + C_4 e^{-cpt}$$

Put X and T in (2)

$$y = \left(C_1 e^{px} + C_2 e^{-px}\right)\left(C_3 e^{cpt} + C_4 e^{-cpt}\right)$$

—— (6)

GW GATEWAY CLASSES
Gateway Classes : 7455 9612 84

CASE -3 : $K = -p^2$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$X = C_1 \cos px + C_2 \sin px$$

From (4)

$$(D^2 + c^2p^2)T = 0$$

$$D^2 + c^2p^2 = 0$$

$$D = \sqrt{-c^2p^2}$$

$$D = \pm icp$$

$$D = 0 \pm icp$$

$$T = C_3 \cos cpt + C_4 \sin cpt$$

Put X and T in (2)

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

GW GATEWAY CLASSES
Gateway Classes : 7455 9612 84

Most suitable solution

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$ — (8)

Apply Boundary condition

(i) At $x = 0$, $y = 0$ ✓

(ii) At $x = l$, $y = 0$

$$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$$

$$\boxed{C_1 = 0}$$

Updated solution

$$y = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)$$ — (9)

$$0 = C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$$ ✓

$$\sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Gateway Classes : 7455 9612 84

Initial conditions

(i) At $t = 0$, $y = f(x)$ ✓✓

(ii) At $t = 0$, $\frac{\partial y}{\partial t} = 0$ ✓✓

Updated solution

$$y = C_2 \sin\frac{n\pi}{l}x\left(C_3\cos\frac{cn\pi}{l}t + C_4\sin\frac{cn\pi}{l}t\right)$$

$$\frac{\partial y}{\partial t} = C_2 \sin\frac{n\pi x}{l}\left(-C_3\sin\frac{cn\pi t}{l}\times\frac{cn\pi}{l} + C_4\cos\frac{cn\pi t}{l}\times\frac{cn\pi}{l}\right)$$

$$0 = C_2 \sin\frac{n\pi x}{l}\left(0 + C_4\times\frac{cn\pi}{l}\right)$$

$$0 = C_2 C_4 \frac{cn\pi}{l}\sin\frac{n\pi x}{l}$$

$$\Rightarrow C_4 = 0$$

Updated solution

$$y = C_2 \sin\frac{n\pi x}{l} C_3\cos\frac{cn\pi t}{l}$$

Gateway Classes : 7455 961284

$$y = (C_2 C_3) \sin\frac{n\pi x}{l} \cos\frac{nc\pi t}{l}$$

Most General solution

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{nc\pi t}{l}$$

At $t = 0$, $y = f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$b_n = ?$

- Comparison
- Fourier series (Half range sine series)

Gateway Classes : 7455 961284

Engineering Mathematics by Gulshan Sir

DPP- 3

Topic : Solution of one Dimensional Wave Equation

Gateway Classes : 7455 9612 84

Q.1. ***Derive one dimensional wave equation.***

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (Part – 2)
- PYQ
- DPP

Gateway Classes : 7455 9612 84

Q.1 : Find the deflection $y(x,t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x, \quad 0 \le x \le 1$$

(G.B.T.U. 2013)

The equation of string

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{---} \quad (1)$$

length of string

$l = 1$ unit

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = 1$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $\frac{\partial y}{\partial t} = 0$ IC-1

At $t = 0$, $y = \sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x$ IC-2

Gateway Classes : 7455 961284

Most suitable solution

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad -(8)$$

Apply BC-1

$$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$$

$$C_1 = 0$$

Updated solution

$$y = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt) \quad -(9)$$

Apply BC-2

$$0 = C_2 \sin p (C_3 \cos cpt + C_4 \sin cpt)$$

$$\Rightarrow \sin p = 0$$

$$\sin p = \sin n\pi$$

$$p = n\pi$$

Gateway Classes : 7455 961284

Updated solution

$$y = C_2 \sin n\pi x \left(C_3 \cos \frac{n\pi c}{} t + C_4 \sin \frac{n\pi c}{} t \right) \quad \text{— (10)}$$

$$\frac{\partial y}{\partial t} = C_2 \sin n\pi x \left(-C_3 \times n\pi c \sin n\pi c t + C_4 \times n\pi c \cos n\pi c t \right)$$

Apply I C – 1

$$0 = C_2 \sin n\pi x \left(0 + n\pi c C_4 \right)$$

$$0 = C_2 C_4 n\pi c \sin n\pi x$$

$$\Rightarrow C_4 = 0$$

Gateway Classes : 7455 96 12 84

Updated solution

$$y = C_2 \sin n\pi x \times C_3 \cos n\pi c t$$

$$y = (C_2 C_4) \sin n\pi x \cos n\pi c t$$

Most general solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cos n\pi c t \quad \text{(1)}$$

Apply I C-2

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x = b_1 \sin \pi x + b_2 \sin 2\pi x + b_3 \sin 3\pi x + b_4 \sin 4\pi x + b_5 \sin 5\pi x + \cdots$$

By comparison

Gateway Classes: 7455 9612 84

$b_1 = 1$

$b_2 = 0$

$b_3 = \frac{1}{3}$

$b_4 = 0$

$b_5 = \frac{1}{5}$

From (11)

$$y(x,t) = b_1 \sin\pi x \cos\pi ct + b_2 \sin 2\pi x \cos 2\pi ct + b_3 \sin 3\pi x \cos 3\pi ct + b_4 \sin 4\pi \cos 4\pi ct + b_5 \sin 5\pi x \cos 5\pi ct$$

$$y(x,t) = \sin\pi x \cos\pi ct + \frac{1}{3}\sin 3\pi x \cos 3\pi ct + \frac{1}{5}\sin 5\pi x \cos 5\pi ct$$

Gateway Classes : 7455 9612 84

Q.2. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$.

Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x, t) = A \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}$$

(AKTU-2018, 2013; UKTU-2011, 2012,2022)

The equation of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

length of string

$l = l$

Boundary conditions

At $x = 0$, $y = 0$ Bc-1

At $x = l$, $y = 0$ Bc-2

Initial conditions

At $t = 0$, $\frac{\partial y}{\partial t} = 0$ Ic-1

At $t = 0$, $y = A \sin \frac{\pi x}{l}$ Ic-2

Gateway Classes: 7455 9012 84

Most Suitable solution

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (8)}$$

Apply Bc-1

$$0 = C_1(C_3 \cos cpt + C_4 \sin cpt)$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$y = C_2 \sin px\,(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (9)}$$

Apply Bc-2

$$0 = C_2 \sin pl\,(C_3 \cos cpt + C_4 \sin cpt)$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Gateway Classes : 7455 96[illegible] 84

Updated solution

$$y = C_2 \sin\frac{n\pi}{l}x\left(C_3\cos\frac{n\pi ct}{l} + C_4\sin\frac{n\pi ct}{l}\right) \quad \text{—— (10)}$$

$$\frac{\partial y}{\partial t} = C_2\sin\frac{n\pi x}{l}\left(-C_3\times\frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + C_4\times\frac{n\pi c}{l}\cos\frac{n\pi ct}{l}\right)$$

Apply IC-1

$$0 = C_2\sin\frac{n\pi x}{l}\times C_4\times\frac{n\pi c}{l}$$

$$\Rightarrow C_4 = 0$$

Gateway Classes : 7453961284

Updated solution

$$y = C_2 \sin\frac{n\pi x}{l} \times C_3 \cos\frac{n\pi ct}{l}$$

$$y = (C_2 C_4) \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

Most General solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

└ (11)

Apply IC-2

$$A \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$A \sin\frac{\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \cdots$$

on comparson

$$b_1 = A$$

Gateway Classes : 7455961284

From (11)

$$y(x,t) = b_1 \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l} + - - -$$

$$\boxed{y(x,t) = A \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l}}$$

Hence proved

Gateway Classes : 7455 9612 84

Engineering Mathematics by Gulshan Sir

DPP- 4

Topic : Solution of one Dimensional Wave Equation (Part -2)

Gateway Classes : 7455 9619 84

Q.1. Find the deflection of the vibrating string which is fixed at the ends $x = 0$ and $x = 2$ and the motion is started by displacing the string into the form $sin^3\left(\frac{\pi x}{2}\right)$ and releasing it with zero initial velocity at $t = 0$ *(U.P.T.U. 2014)*

Q.2 A string *is* stretched and fastened to two points distance l apart. Find the displacement $y(x, t)$ at any point at a distance x from one end at time t given that:

$$siny(x, 0) = A\, sin\left(\frac{2\pi x}{l}\right)$$

(M.T.U. 2013)

Q.3 A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0\, sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.

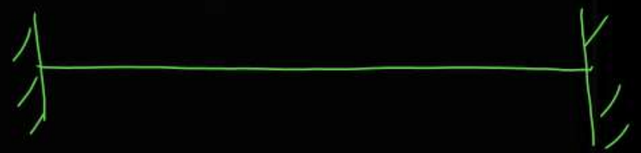
(GBTU-2011,2020)

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (Part – 3)
- PYQ
- DPP



Gateway Classes : 7455 9612 84

Q.1 : A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially ($t = 0$) at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity ($t = 0$) $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t. (MTU-2011)

The equation of String

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——} \quad (1)$$

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = l$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $y = 0$ IC-1

At $t = 0$, $\frac{dy}{dt} = \lambda x(l - x)$ IC-2

Let $y = XT$ —— (2)

Gateway Classes : 7455 9612 84

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2} = XT''$$

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2} = TX''$$

Put these values in (1)

$$XT'' = c^2 TX''$$

$$\frac{XT''}{XT} = c^2\frac{TX''}{XT}$$

$$\frac{T''}{T} = c^2\frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2}\times\frac{T''}{T} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad -(3)$$

$$\frac{1}{c^2}\times\frac{T''}{T} = K$$

$$T'' = c^2KT$$

$$T'' - c^2KT = 0$$

$$D^2T - c^2KT = 0$$

$$(D^2 - c^2K)T = 0 \quad -(4)$$

Gateway Classes : 7455 961284

CASE–1 When $K = 0$

$$D^2X = 0 \qquad D^2T = 0$$

$$D^2 = 0 \qquad D^2 = 0$$

$$D = 0, 0 \qquad D = 0, 0$$

$$CF = C_1 + C_2x \qquad CF = C_3 + C_4t$$

$$PI = 0 \qquad PI = 0$$

$$X = C_1 + C_2x \qquad T = C_3 + C_4t$$

Put X and T in (2)

$$\boxed{y = (C_1 + C_2x)(C_3 + C_4t)} \quad \text{— (5)}$$

CASE –2 : When $K = p^2$

$$(D^2 - p^2)X = 0 \qquad (D^2 - c^2p^2)T = 0$$

$$D^2 - p^2 = 0 \qquad D^2 - c^2p^2 = 0$$

$$D = \pm p \qquad D = \pm cp$$

$$CF = C_1e^{px} + C_2e^{-px} \qquad CF = C_3e^{cpt} + C_4e^{-cpt}$$

$$PI = 0 \qquad PI = 0$$

$$X = C_1e^{px} + C_2e^{-px} \qquad T = C_3e^{cpt} + C_4e^{-cpt}$$

Put X and T in (2)

$$\boxed{y = (C_1e^{px} + C_2e^{-px})(C_3e^{cpt} + C_4e^{-cpt})}$$

— (6)

Gateway Classes: 7455 9612 84

CASE-3: When $k = -p^2$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D = \sqrt{-p^2}$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$X = C_1 \cos px + C_2 \sin px$$

$$(D^2 + c^2p^2)T = 0$$

$$D^2 + c^2p^2 = 0$$

$$D = \sqrt{-c^2p^2}$$

$$D = \pm icp$$

$$D = 0 \pm icp$$

$$T = C_3 \cos cpt + C_4 \sin cpt$$

Put X and T in (2)

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (7)}$$

Most suitable solution from solutions (5), (6) and (7)

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (8)}$$

Gateway Classes : 7455 961284

Apply Bc-1

$$0 = C_1(C_3\cos cpt + C_4\sin cpt)$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$y = C_2\sin px\,(C_3\cos cpt + C_4\sin cpt) \quad \text{--- (9)}$$

Apply Bc-2

$$0 = C_2\sin pl\,(C_3\cos cpt + C_4\sin cpt)$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Updated solution

$$y = C_2\sin\frac{n\pi}{l}x\left(C_3\cos\frac{n\pi ct}{l} + C_4\sin\frac{n\pi ct}{l}\right) \quad \text{--- (10)}$$

Apply Ic-1

Gateway Classes: 7455 961284

$$0 = C_2 \sin\frac{n\pi x}{l} \times C_3$$

$$\Rightarrow C_3 = 0$$

Updated solution

$$y = C_2 \sin\frac{n\pi x}{l} \times C_4 \sin\frac{n\pi ct}{l} \quad \text{(11)}$$

Most general solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l} \quad \text{(12)}$$

Where $b_n = C_2 C_4$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times \cos\frac{n\pi ct}{l} \times \frac{n\pi c}{l}$$

Apply IC-2

$$-\ln(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times \frac{n\pi c}{l}$$

$$\ln(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{l}$$

GW GATEWAY CLASSES
Gateway Classes : 7455961284

$$\lambda n(l-n) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi n}{l}$$

Half range sine series

$$f(n) = \sum b_n \sin \frac{n\pi n}{l}$$

Where

$$b_n = \frac{2}{l} \int_0^l f(n) \sin \frac{n\pi n}{l} dn$$

$$\frac{\pi c}{l} n b_n = \frac{2}{l} \int_0^l \lambda n(l-n) \sin \frac{n\pi n}{l} dn$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l} \int_0^l (ln - n^2) \sin \frac{n\pi n}{l} dn$$

I II

$$\int uv\,dn = uv_1 - u'v_2 + u''v_3 - \cdots$$

Formula ↗

Gateway Classes. 7455 9612 84

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[(ln - n^2)\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (l-2n)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + (-2)\frac{\left(\cos\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[0 - 0 - \frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} - \left(0 - 0 - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[-\frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} + \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

Gateway Classes : 7455 9612 84

$$\frac{\pi c}{\ell} n b_n = \frac{2\lambda}{\ell \times \left(\frac{n\pi}{\ell}\right)^3}\left[2 - 2\cos n\pi\right]$$

$$b_n = \frac{2\lambda \times \ell^3}{\pi c n \times n^3\pi^3} \times 2\left(1 - \cos n\pi\right)$$

$$b_n = \frac{4\lambda\ell^3}{c n^4 \pi^4}\left[1 - (-1)^n\right]$$

From (11)

$$y = \sum_{n=1}^{\infty} \frac{4\lambda\ell^3}{c n^4 \pi^4}\left[1 - (-1)^n\right] \sin\frac{n\pi x}{\ell} \sin\frac{n\pi c t}{\ell}$$

Gateway Classes : 7455 9612 84

$$y(x,t) = \frac{4\lambda l^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left[1-(-1)^n\right] \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

If n is even

$$y(x,t) = 0$$

If n is odd

$$y(x,t) = \frac{8\lambda l^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

GW GATEWAY CLASSES
GatewayClasses : 7455 9612 84

DPP- 5

Topic : Solution of one Dimensional Wave Equation

Gateway Classes : 7455 9612 84

Q.1. **Show how the wave equation $c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ *Can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between* $x = 0$ *and* $x = l$ *are given by* $y = f(x)$ *and* $\frac{\partial y}{\partial t} = g(x)$*, determine the constants in the series solution. .* *(UPTU 2014)***

Q.2 **A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, μ is a constant and then released. Find the displacement $y(x, t)$ of any point x of the string at any time $t > 0$.** *(M.T.U. 2015)*

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

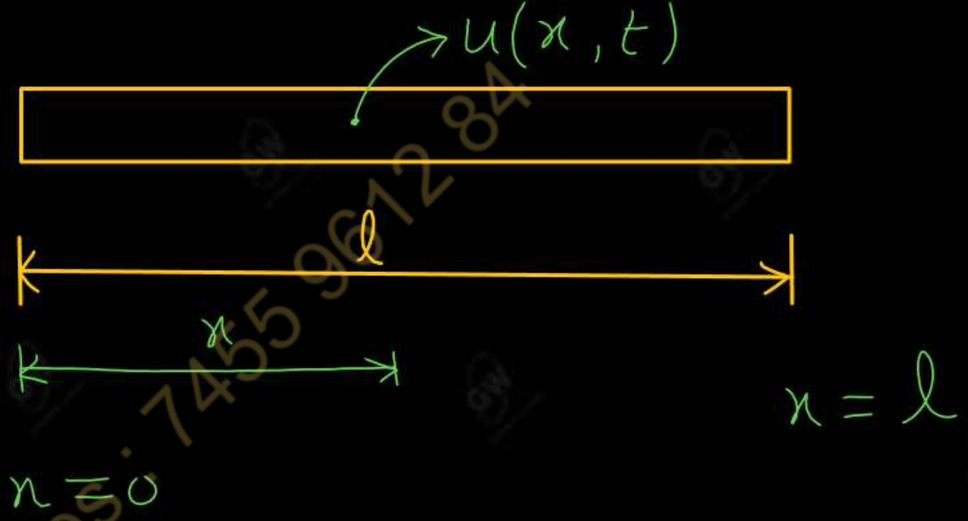
Today's Target

- One Dimensional Heat Equation (Part – 1)
- PYQ
- DPP

Gateway Classes : 7455 9612 84

One Dimensional Heat Equation

Consider the flow of heat in a homogeneous rod of length l and variation of temperature with position and time.



Assumptions

(i) Rod has uniform cross-section

(ii) Rod has constant density

(iii) Lateral surface of rod is insulated

(iv) Heat flow only in one direction

(v) The rod is sufficiently thin so that the temperature is same at all points of any cross-section area of rod

Let $u(x, t)$ be the temperature distribution of the cross-section at the point x and at any time t

Mathematically

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This partial differential equation is known as one-dimensional Heat equation

Gateway Classes : 7455 9612 84

Q. 1:- *Find the solution of One – dimensional heat equation by the method of separation of variables*

One - Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

Let $u = XT$ —— (2)

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = TX''$$

Put these values in (1)

$$XT' = c^2 TX''$$

Divide by XT

$$\frac{XT'}{XT} = \frac{c^2 TX''}{XT}$$

$$\frac{T'}{T} = c^2 \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2} \times \frac{T'}{T} = K \text{ (say)}$$

Gateway Classes : 7455 3612 64

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{—(3)}$$

$$\frac{1}{c^2} \times \frac{T'}{T} = K$$

$$T' = c^2 KT$$

$$DT - c^2 KT = 0$$

$$(D - c^2 K)T = 0 \quad \text{—(4)}$$

CASE-1: When $K = 0$

$$D^2X = 0$$

$$D^2 = 0$$

$$D = 0, 0$$

$$X = c_1 + c_2 x$$

$$DT = 0$$

$$D = 0$$

$$T = c_3 e^{0t}$$

$$T = c_3$$

Put x and T in (2)

Gateway Classes : 7455 96 12 84

$$u(x,t) = (C_1 + C_2 x) C_3$$

$$u(x,t) = C_1 C_3 + C_2 C_3 x$$

$$\boxed{u(x,t) = a + bx} \quad \text{—(5)}$$

Time-independent solution

$\Rightarrow$ steady-state solution

CASE-2: When $K = P^2$

$$(D^2 - P^2)X = 0$$

$$D^2 - P^2 = 0$$

$$D^2 = P^2$$

$$D = \pm P$$

$$X = C_1 e^{Px} + C_2 e^{-Px}$$

$$(D - P^2 C^2)T = 0$$

$$D - P^2 C^2 = 0$$

$$D = P^2 C^2$$

$$T = C_3 e^{P^2 C^2 t}$$

Put these values in (2)

$$\boxed{u(x,t) = (C_1 e^{Px} + C_2 e^{-Px}) C_3 e^{P^2 C^2 t}}$$

(6)

CASE-3: When $K = -P^2$

$$(D^2 + P^2)X = 0$$

$$D^2 + P^2 = 0$$

$$D = \sqrt{-P^2}$$

$$D = \pm iP$$

GW GATEWAY CLASSES
Gateway Classes: 7455 961 284

$D = 0 + iP$

$X = C_1 \cos Px + C_2 \sin Px$

$(D + c^2P^2)T = 0$

$D + c^2P^2 = 0$

$D = -c^2P^2$

$T = C_3 e^{-c^2P^2t}$

Put these values in (2)

$$u(x,t) = (C_1 \cos Px + C_2 \sin Px) C_3 e^{-c^2P^2t} \quad — (7)$$

Most suitable solution

$$u(x,t) = (C_1 \cos Px + C_2 \sin Px) C_3 e^{-c^2P^2t} \quad — (8)$$

Steady state solution

$u(x,t) = a + bx$

Gateway Classes 7455961284

comple solution

$$u(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2 p^2 t} + a + bx$$

$$u(x,t) = u_t(x,t) + u_s(x)$$

Transient solution

steady state solution

Gateway Classes : 7455 961284

On the Basis of Boundary conditions

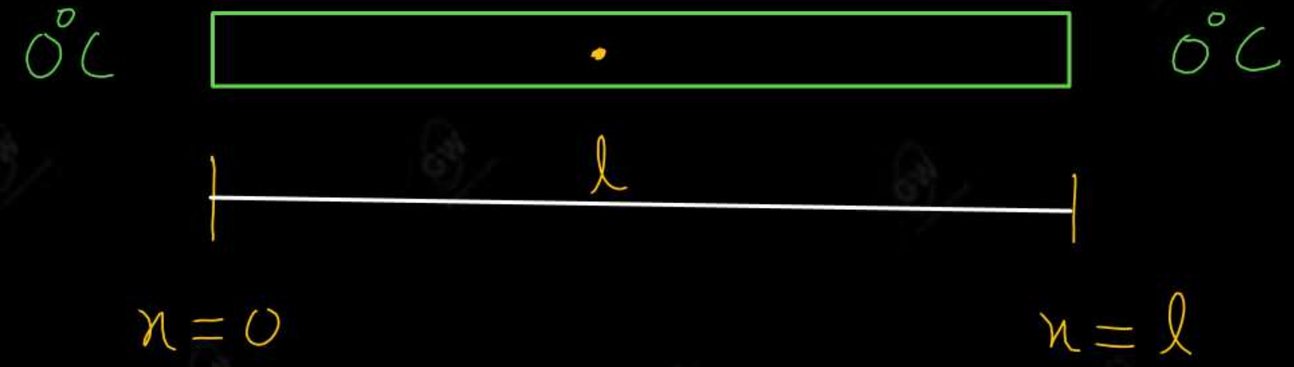
Type -1: When both ends of rod are at 0^oC

Type -2: When both ends of rod are perfectly insulated

Type -3: When both ends of rod are kept at different temperature.

Gateway Classes : 7455 9612 84

Type -1 : When both ends of rod are at 0^oC



$$u(x,t) = u_s(x) + u_t(x,t)$$

Steady state | Transient state

Boundary Conditions

$$At\ x = 0,\ u = 0 \Rightarrow u(0,t) = 0^oC \quad BC-1$$

$$At\ x = l,\ u = 0 \Rightarrow u(l,t) = 0^oC \quad BC-2$$

$u_s(x) = a + bx$

Apply BC-1

$0 = a + 0$

$a = 0$

$u_s(x) = bx$

$0 = bl$

$b = 0$

$\Rightarrow \boxed{u_s(x) = 0}$

$u(x,t) = u_t(x,t)$

$$\boxed{u(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2p^2t}}$$

Gateway Classes : 7455 9612 84

Q. 1:- *Find the temperature in a bar of length 2 m whose ends are kept at zero and lateral surface insulated if the initial temperature is* $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$. (U.P.T.U. 2015)

One-Dimensional Heat equation

$$\frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——— (1)}$$

Length of rod $(l) = 2m$

Boundary conditions

At $x = 0$, $u = 0$ B(-1

At $x = 2$, $u = 0$ B(-2

Initial condition

At $t = 0$, $u = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Most Suitable solution

$$u(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2 p^2 t}$$

└ (8)

Apply B(-1

$$0 = C_1 C_3 e^{-c^2 p^2 t}$$

Gateway Classes: 7455 961284

$\Rightarrow C_1 = 0$

Updated solution

$$u(x,t) = C_2 \sin px \times C_3 e^{-c^2 p^2 t}$$ — (9)

Apply BC-2

$$0 = C_2 \sin 2p \times C_3 e^{-c^2 p^2 t}$$

$\Rightarrow \sin 2p = 0$

$\sin 2p = \sin n\pi$

$2p = n\pi$

$p = \frac{n\pi}{2}$

Updated solution

$$u(x,t) = C_2 \sin \frac{n\pi}{2} x \times C_3 e^{-\frac{c^2 n^2 \pi^2 t}{4}}$$

$$u(x,t) = C_2 C_3 \sin \frac{n\pi}{2} x \times e^{-\frac{c^2 n^2 \pi^2}{4} t}$$ — (10)

Most general solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x \times e^{-\frac{c^2 n^2 \pi^2 t}{4}}$$ — (11)

Gateway Classes : 7455 961 284

Apply IC

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = b_1 \sin\frac{\pi x}{2} + b_2 \sin\pi x + b_3 \sin\frac{3\pi x}{2} + b_4 \sin 2\pi x + b_5 \sin\frac{5\pi x}{2} + -$$

on comparison

$b_1 = 1$ $\quad b_4 = 0$

$b_2 = 0$ $\quad b_5 = 3$

$b_3 = 0$

Gateway Classes : 7455 9612 84

From (11)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}x \times e^{\frac{-c^2n^2\pi^2t}{4}}$$

$$u(x,t) = b_1 \sin\frac{\pi x}{2} \times e^{\frac{-c^2\pi^2t}{4}} + b_5 \sin\frac{5\pi x}{2} e^{\frac{-c^2 25\pi^2 t}{4}}$$

$$u(x,t) = e^{\frac{-c^2\pi^2t}{4}} \sin\frac{\pi x}{2} + 3e^{\frac{-25c^2\pi^2t}{4}} \sin\frac{5\pi x}{2}$$

Gateway Classes : 7455 961284

Q. 2:- Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$. (G.B.T.U. 2012)

One-Dimensional Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \quad (1)$$

length of rod = 2m

Boundary conditions

At $x = 0$, $u = 0$ B.C-1

At $x = 2m$, $u = 0$ B.C-2

Initial condition

At $t = 0$, $u = f(x) = 100x$

Most suitable solution

$$y(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2 p^2 t}$$

└ (8)

Apply B.C-1

$$0 = C_1 C_3 e^{-c^2 p^2 t}$$

Gateway Classes 7455 961284

$\Rightarrow c_1 = 0$

Updated solution

$$u(x,t) = c_2 \sin px \times c_3 e^{-c^2 p^2 t}$$

(9)

Apply BC-2

$$0 = c_2 \sin 2p \times c_3 e^{-c^2 p^2 t}$$

$\Rightarrow \sin 2p = 0$

$\sin 2p = \sin n\pi$

$2p = n\pi$

$p = \frac{n\pi}{2}$

Updated solution

$$u(x,t) = c_2 \sin \frac{n\pi}{2} x \times c_3 e^{-c^2 p^2 t}$$

(10)

Most General solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x \times e^{-c^2 p^2 t}$$

(11)

Gateway Classes : 7455 961284

Apply I C

$$100x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

Using Half-Range sine-series

$$b_n = \frac{2}{2}\int_0^2 100x \times \sin\frac{n\pi x}{2}\,dx$$

$$b_n = 100\int_0^2 x \sin\frac{n\pi x}{2}\,dx$$

I II

$$b_n = 100\left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^2$$

$$b_n = 100\left[\frac{-2\cos n\pi}{\frac{n\pi}{2}} - 0 - (0-0)\right]$$

$$b_n = 100\left(\frac{-4\cos n\pi}{n\pi}\right)$$

$$b_n = -\frac{400}{\pi}\left(\frac{\cos n\pi}{n}\right)$$

GW GATEWAY CLASSES
Gateway Classes : 7455 96 12 84

Put b_n in (11)

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{-400}{\pi}\right)\left(\frac{\cos n\pi}{n}\right) \sin\frac{n\pi x}{2} \times e^{-c^2p^2t}$$

$$u(x,t) = \frac{-400}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos n\pi}{n}\right) \sin\frac{n\pi x}{2} \times e^{-c^2p^2t}$$

Where $\cos n\pi = (-1)^n$

Gateway Classes: 7455 9612 84

DPP- 6

Topic : One Dimensional Heat Equation

Q.1. A *rod of length l with insulated sides is initially at a uniform temperature* u_0. *Its ends are suddenly cooled to 0°C and are kept* at *that temperature. Find the temperature function* $u(x, t)$. *(U.P.T.U. 2015)*

Q.2. Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $u(0, t) = 0, u(l, t) = 0\ (t > 0)$ and the initial condition $u(x, 0) = x$; l being the length of the bar. *(U.P.T.U. 2015)*

Gateway Classes : 7455 961294

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- One Dimensional Heat Equation (Part – 2)
- PYQ
- DPP

Gateway Classes : 7455 9612 84

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

One Dimensional Heat Equation :

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

On the Basis of Boundary conditions, there are three types of question

Type -1: When both ends of rod are at $0^o C$

Type -2: When both ends of rod are perfectly insulated

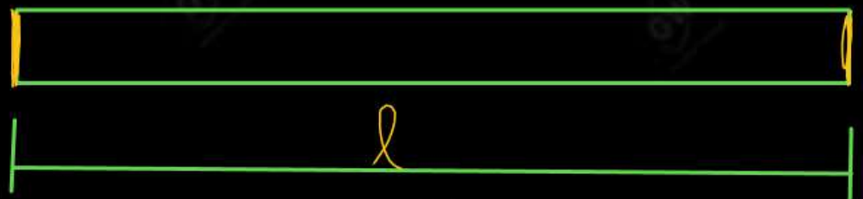
Type -3: When both ends of rod are kept at different temperature.

Gateway Classes : 7455 9612 84

Type -2 : When both ends of rod are perfectly insulated

$\frac{\partial u}{\partial x}$

Since ends of rod are insulated, no heat can transfer from either sides.



$x = 0$ $\qquad$ $x = l$

$$u(x,t) = u_s(x) + u_t(x,t)$$

Steady state $\qquad$ *Transient state*

Boundary Conditions

At $x = 0$, $\frac{\partial u}{\partial x} = 0$

At $x = l$, $\frac{\partial u}{\partial x} = 0$

$u_s = a + bx$

$\frac{\partial u}{\partial x} = 0 + b$

Apply BC-1

$0 = b$

$u_s = a$

$u(x,t) = a + u_t(x,t)$

$$u(x,t) = a + (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2 p^2 t}$$

Gateway Classes : 7455 9612 84

Q. 1:- Find the temperature distribution in a rod of length π which is totally insulated including the ends and the initial temperature distribution is $100 \cos x$. (U.P.T.U. 2015)

one-dimensional Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \quad (1)$$

length of rod $(l) = \pi$

Boundary conditions

At $x = 0$, $\frac{\partial u}{\partial x} = 0$ BC-1

At $x = \pi$, $\frac{\partial u}{\partial x} = 0$ BC-2

Initial condition

At $t = 0$, $u = 100 \cos x$ IC

complete solution of (1)

$$u(x,t) = a + (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t} \quad (8)$$

$$\frac{\partial u}{\partial x} = 0 + (-c_1 p \sin px + c_2 p \cos px) c_3 e^{-c^2 p^2 t}$$

Apply BC-1

$$0 = c_2 p \times c_3 e^{-c^2 p^2 t}$$

Gateway Classes, 7455 9612 84

$\Rightarrow C_2 = 0$

Updated solution

$$u(x,t) = a + (C_1 \cos px)\, C_3 e^{-c^2 p^2 t}$$ — (9)

$$\frac{\partial u}{\partial x} = 0 + (-C_1 p \sin px)\, C_3 e^{-c^2 p^2 t}$$

Apply BC-2

$$0 = -C_1 p \sin \pi p \times C_3 e^{-c^2 p^2 t}$$

$\Rightarrow \sin p\pi = 0$

$\sin p\pi = \sin n\pi$

$p\pi = n\pi$

$p = n$

Updated solution

$$u = a + (C_1 \cos nx)\, C_3 e^{-c^2 n^2 t}$$ — (10)

Most-General solution

$$u = a + \sum_{n=1}^{\infty} b_n \cos nx \times e^{-c^2 n^2 t}$$ — (11)

Gateway Classes : 7455 961284

Apply I C

$$100\cos x = a + \sum_{n=1}^{\infty} b_n \cos nx$$

$$100\cos x = a + b_1 \cos x + b_2 \cos 2x + \dots$$

By comparison

$a = 0$

$b_1 = 100$

$b_2 = 0$

$b_3 = 0$

From (11)

$$u(x,t) = a + \sum_{n=1}^{\infty} b_n \cos nx \times e^{-c^2 n^2 t}$$

$$= a + b_1 \cos x \times e^{-c^2 t} + b_2 \cos 2x \times e^{-c^2 x^2 x t}$$

$$u(x,t) = 0 + 100 \cos x \times e^{-c^2 t}$$

$$\boxed{u(x,t) = 100 e^{-c^2 t} \cos x}$$

Gateway Classes : 7455 9612 84

Q.1. The temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Assuming the initial temperature distribution as $u(x, 0) = f(x) = \cos 2x$, find the temperature distribution at any instant of time. **(M.T.U. 2011)**

Gateway Classes : 7455 9612 84

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- One Dimensional Heat Equation (Part – 3)
- PYQ
- DPP

Gateway Classes : 7455 9612 84

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

One Dimensional Heat Equation : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

On the Basis of Boundary conditions, there are three types of question

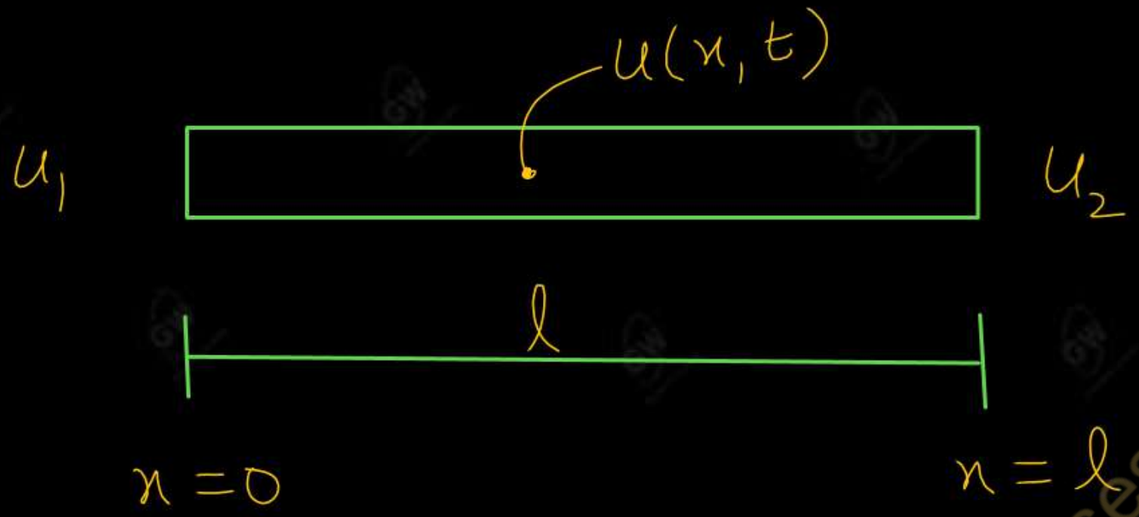
Type -1: When both ends of rod are at $0^o C$

Type -2: When both ends of rod are perfectly insulated

Type -3: When both ends of rod are kept at different temperature.

Gateway Classes 7455 9612 84

Type -3 : When both ends of rod are kept at different temperature.



$$u_s = a + bx$$

$$u_t = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}$$

complete solution

$$u(x,t) = u_s(x) + u_t(x,t)$$

$u_s(x)$: Steady state

$u_t(x,t)$: Transient state

Gateway Classes : 7455 961 284

Q. 1: A bar of 10 cm length with insulated sides A and B are kept and 20°C and 40°C respectively until steady state conditions prevail. The temperature at A is then suddenly varied to 50°C and the same instant at B, lowered at 10°C. Find the subsequent temperature at any point of the bar at any time. *(A.K.T.U. 2017)*

One-dimensional heat equation

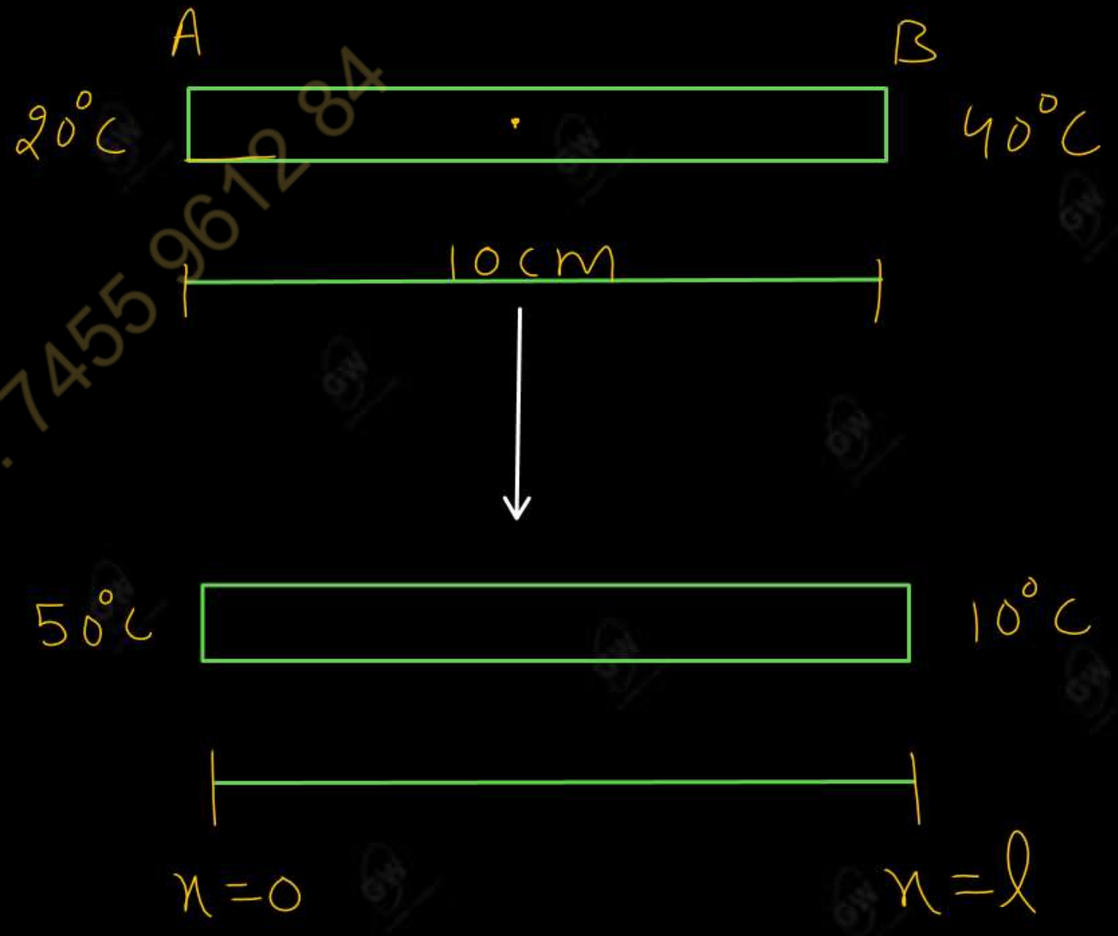
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{---} \quad (1)$$

length of bar $(l) = 10\,cm$

Boundary conditions

At $x = 0$, $u = 50^\circ$ B.C-1

At $x = 10\,cm$, $u = 10^\circ C$ B.C-2



Gateway Classes : 7455 9612 84

Initial condition

$$u = T_A + \left(\frac{T_B - T_A}{10}\right)x$$

$$u = 20 + \left(\frac{40-20}{10}\right)x$$

$$u = 20 + 2x$$

At $t = 0$, $u = 20 + 2x$

Complete solution of (1)

$$u(x,t) = u_s(x) + u_s(x,t)$$

$$u(x,t) = (a+bx) + (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2 p^2 t} \quad \text{— (8)}$$

Steady state solution

$$u_s(x) = T_A + \left(\frac{T_B - T_A}{10}\right)x$$

$$= 50 + \frac{(10-50)}{10}x$$

$$\boxed{u_s(x) = 50 - 4x}$$

Put in (8)

Gateway Classes : 1455 9612 84

$$u(x,t) = (50-4x) + (C_1 \cos px + C_2 \sin px) C_3 e^{-c^2p^2t} \quad \text{— (9)}$$

Apply BC-1

$$\cancel{50} = \cancel{50} + C_1 \times C_3 e^{-c^2p^2t}$$

$$0 = C_1 C_3 e^{-c^2p^2t}$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$u(x,t) = (50-4x) + C_2 \sin px \times C_3 e^{-c^2p^2t} \quad \text{— (10)}$$

Apply BC-2

$$10 = 50 - 40 + C_2 \sin 10p \times C_3 e^{-c^2p^2t}$$

$$\cancel{10} = \cancel{10} + C_2 \sin 10p \times C_3 e^{-c^2p^2t}$$

$$0 = C_2 \sin 10p \times C_3 e^{-c^2p^2t}$$

$$\Rightarrow \sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

Gateway Classes : 7455 9012 84

Updated solution

$$u(x,t) = (50-4x) + c_2 \sin\frac{n\pi}{10}x \times c_3 e^{-\frac{c^2 n^2 \pi^2}{100}t} \quad \text{—— (11)}$$

Most general solution

$$u(x,t) = (50-4x) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x \times e^{-\frac{c^2 n^2 \pi^2}{100}t} \quad \text{—— (12)}$$

Apply initial condition

$$20+2x = 50-4x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x \times 1$$

Gateway Classes. 7455 9612 84

$$-30+6x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Apply Half range sine series

Formula

$$f(x) = \sum b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{10}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

I II

$$b_n = \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$b_n = \frac{1}{5}\left[\frac{-30\cos n\pi}{\frac{n\pi}{10}} - 0 - \left(\frac{-30(-1)}{\frac{n\pi}{10}} - 0\right)\right]$$

Gateway Classes : 7498961284

$$b_n = \frac{1}{5}\left[\frac{-300\cos n\pi}{n\pi} - \frac{300}{n\pi}\right]$$

$$b_n = \frac{-300}{5 n\pi}\left(\cos n\pi + 1\right)$$

$$b_n = -\frac{60}{n\pi}\left(1 + \cos n\pi\right)$$

Put b_n in (12)

$$u(x,t) = (50 - 4x) + \sum_{n=1}^{\infty} \frac{-60}{n\pi}\left(1 + \cos n\pi\right) \times \sin\frac{n\pi x}{10} \times e^{-\frac{c^2 n^2 \pi^2}{100} t}$$

Gateway Classes : 7455 9612 84

$$u(x,t) = (50-4x) - \frac{60}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(1+\cos n\pi)\sin\frac{n\pi x}{10}\, e^{-\frac{c^2 n^2 \pi^2}{100}t}$$

Gateway Classes : 7455 9612 84

Gateway Classes : 7455 961284

Q.1. **(i) An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.** **[G.B.T.U. (A.G.) 2011, U.K.T.U. 2011]**

(ii) Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Two Dimensional Heat Equation (Laplace Equation) Part-1
- PYQ
- DPP

Gateway Classes : 7455 9612 84

Application of partial differential equations

1. Solution of one Dimensional Wave Equation 3L
2. Solution of one Dimensional Heat Equation 3L
3. Solution of two Dimensional heat Equation 2L

Method of Separation of Variables 2L

Gateway Classes : 7455 9612 84

Two Dimensional Heat Equation (Laplace Equation)

(i) Two Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(ii) Two Dimensional Heat Equation in steady state (Laplace Equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\hookrightarrow \frac{\partial u}{\partial t} = 0$

known as Laplace equation

Gateway Classes 7455 9612 84

Solution of Laplace equation in two dimension by using Separation of Variables Method.

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} (1)$$

Let $u(x, y) = XY$ —— (2)

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{d^2X}{dx^2} = YX''$$

$$\frac{\partial^2 u}{\partial y^2} = X\frac{d^2Y}{dy^2} = XY''$$

Put these values in (1)

$$YX'' + XY'' = 0$$

Divide by XY

$$\frac{YX''}{XY} + \frac{XY''}{XY} = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{——} (3)$$

$$\frac{-Y''}{Y} = K$$

Gateway Classes : 7455 961284

$-Y'' = KY$

$Y'' + KY = 0$

$D^2Y + KY = 0$

$(D^2 + K)Y = 0$ — (4)

CASE-1 : When $K = 0$

$(D^2 - K)X = 0$

$D^2 X = 0$

$D^2 = 0$

$D = 0, 0$

$X = C_1 + C_2 x$

$(D^2 + K)Y = 0$

$D^2 Y = 0$

$D^2 = 0$

$D = 0, 0$

$Y = C_3 + C_4 y$

Put x and y in (2)

$$u(x, y) = (C_1 + C_2 x)(C_3 + C_4 y)$$

(5)

GW GATEWAY CLASSES

Gateway Classes : 7455 9612 84

CASE-2: When $k = p^2$

$$(D^2 - k)X = 0$$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D^2 = p^2$$

$$D = \pm p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$(D^2 + k)Y = 0$$

$$(D^2 + p^2)Y = 0$$

$$D^2 + p^2 = 0$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$Y = c_3 \cos py + c_4 \sin py$$

Put X and Y in (2)

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

(6)

CASE-3: When $k = -p^2$

$$(D^2 - k)X = 0$$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D^2 = -p^2$$

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$D = \sqrt{-p^2}$

$D = 0 \pm ip$

$X = c_1 \cos px + c_2 \sin px$

$(D^2 + k)Y = 0$

$(D^2 - p^2)Y = 0$

$D^2 - p^2 = 0$

$D^2 = p^2$

$D = \pm p$

$Y = c_3 e^{py} + c_4 e^{-py}$

Put these values in (2)

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

(7)

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Solution of Laplace Equation

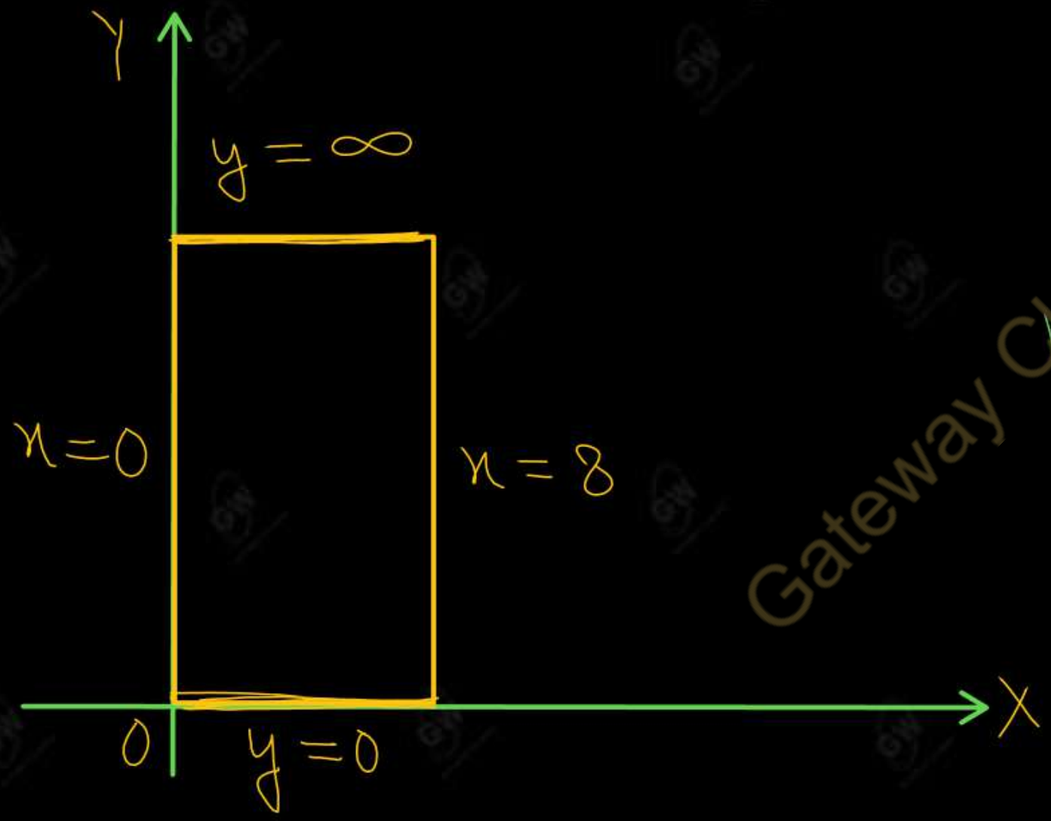
$$u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y) \quad \text{—— (5)} \qquad \times \qquad k = 0$$

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{— (6)} \qquad k = p^2$$

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{— (7)} \qquad k = -p^2$$

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Q. 1: A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}, 0 < x < 8$ *while the two long edges* $x = 0$ *and* $x = 8$ *as well as the other short edge are kept at* 0^0 *C, show that the steady state temperature at any point of the plate is given by* $u(x, y) = 100\, e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}$.



Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{---} (1)$$

where $u(x, y)$ is steady state temp of plate

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Under the Boundary condition

$u(0,y) = 0$ BC-1

$u(8,y) = 0$ BC-2

$u(x,\infty) = 0 \Rightarrow \lim_{y\to\infty} u(x,y) = 0$ BC-3

$u(x,0) = 100 \sin\frac{\pi x}{8}$ BC-4

solution of Laplace equation

$$u(x,y) = (C_1 + C_2 x)(C_3 + C_4 y) \quad -(5)$$

$$u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad -(6)$$

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad -(7)$$

Most suitable solution

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad -(8)$$

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Apply BC-1

$$0 = C_1(C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$u(x,y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (9)}$$

Apply BC-2

$$0 = C_2 \sin 8p\,(C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow \sin 8p = 0$$

$$\sin 8p = n\pi$$

$$8p = n\pi$$

$$p = \frac{n\pi}{8}$$

Updated solution

$$u(x,y) = C_2 \sin \frac{n\pi}{8} x \left(C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}}\right) \quad \text{— (10)}$$

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Apply BC-3

$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} C_2 \sin\frac{n\pi x}{8}\left(C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}}\right)$$

$$0 = C_2 \sin\frac{n\pi x}{8}\left[\lim_{y\to\infty} C_3 e^{\frac{n\pi y}{8}} + \lim_{y\to\infty} C_4 e^{-\frac{n\pi y}{8}}\right]$$

$$\Rightarrow C_3 = 0$$

Updated solution

$$u(x,y) = C_2 \sin\frac{n\pi x}{8} \times C_4 e^{-\frac{n\pi y}{8}} \quad \text{— (11)}$$

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Most general solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8} e^{-\frac{n\pi y}{8}}$$

Apply BC-4

$$100 \sin\frac{\pi x}{8} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8}$$

$$100 \sin\frac{\pi x}{8} = b_1 \sin\frac{\pi x}{8} + b_2 \sin\frac{2\pi x}{8} + \cdots$$

on comparison

$$b_1 = 100$$

$$n = 1$$

$$u(x,y) = b_1 \sin\frac{\pi x}{8} e^{-\frac{\pi y}{8}}$$

$$u(x,y) = 100 \sin\frac{\pi x}{8} e^{-\frac{\pi y}{8}}$$

$$\boxed{u(x,y) = 100 e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}}$$

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DPP- 9

Topic : Two Dimensional Heat Equation (Laplace Equation)

Part-1

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Q.1. Solve temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < \pi, 0 < y < \pi,$$ which satisfies the conditions :

$u(0,y) = u(\pi, y) = u(x, \pi) = 0$ and $u(x, 0) = sin^2 x$ (U.K.T.U. 2011)

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Two Dimensional Heat Equation (Laplace Equation) Part-2
- PYQ
- DPP

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Application of partial differential equations

1. Solution of one Dimensional Wave Equation 3L
2. Solution of one Dimensional Heat Equation 3L
3. Solution of two Dimensional heat Equation 2L

Method of separation of variables 2L

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Two Dimensional Heat Equation (Laplace Equation)

(i) Two Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(ii) Two Dimensional Heat Equation in steady state (Laplace Equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

known as Laplace equation

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Solution of Laplace equation in two dimension by using Separation of Variables Method.

(1) $$u(x,y) = (C_1 + C_2 x)(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (5)}$$

(2) $$u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad \text{— (6)}$$

(3) $$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (7)}$$

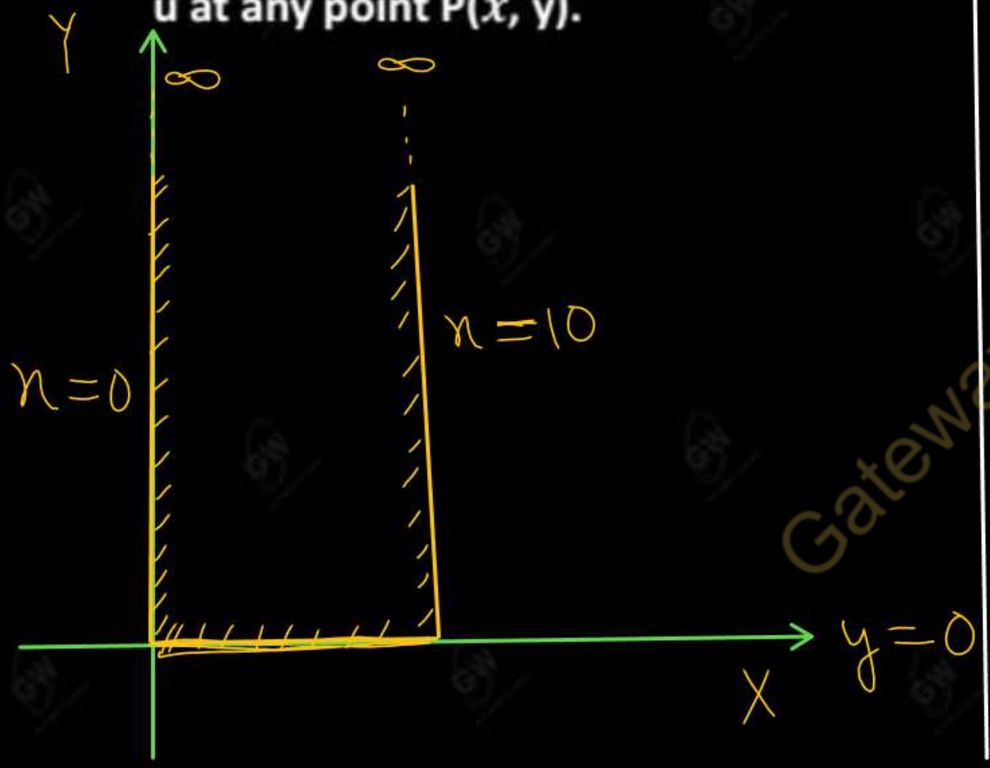
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Q.1 A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. IF the temperature along the short edge y = 0 is given by

And $u(x, y) = \begin{cases} 20x, & 0 < x \leq 5 \\ 20(10 - x), & 5 < x < 10 \end{cases}$

And the two long edges x = 0 and x = 10 as well as other short edge are kept at 0°C. Find the temperature u at any point P(x, y).

(A.K.T.U. 2018)



Let $u(x, y)$ be the steady state temp. given by Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

Under the Boundary conditions

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$u(0,y) = 0$ BC-1

$u(10,y) = 0$ BC-2

$\lim_{y\to\infty} u(x,y) = 0$ BC-3

$$u(x,0) = \begin{cases} 20x & 0<x\le 5 \\ 20(10-x) & 5<x<10 \end{cases}$$ BC-4

Solution of eqn (1)

$$u(x,y) = (c_1\cos px + c_2\sin px)(c_3e^{py} + c_4e^{-py})$$ — (7)

Apply BC-1

$$0 = c_1(c_3e^{py} + c_4e^{-py})$$

$$\Rightarrow c_1 = 0$$

Updated solution

$$u(x,y) = c_2\sin px(c_3e^{py} + c_4e^{-py})$$ — (8)

Apply BC-2

$$0 = c_2\sin 10p(c_3e^{py} + c_4e^{-py})$$

$$\Rightarrow \sin 10p = 0$$

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$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

Updated solution

$$u(x,y) = C_2 \sin\frac{n\pi x}{10}\left(C_3 e^{\frac{n\pi y}{10}} + C_4 e^{\frac{-n\pi y}{10}}\right) \quad \text{— (9)}$$

Apply BC-3

$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} C_2 \sin\frac{n\pi x}{10}\left(C_3 e^{\frac{n\pi y}{10}} + C_4 e^{\frac{-n\pi y}{10}}\right)$$

$$0 = C_2 \sin\frac{n\pi x}{10}\left[\lim_{y\to\infty} C_3 e^{\frac{n\pi y}{10}} + \lim_{y\to\infty} C_4 e^{\frac{-n\pi y}{10}}\right]$$

$$0 = C_2 \sin\frac{n\pi x}{10}\left[\lim_{y\to\infty} C_3 e^{\frac{n\pi y}{10}}\right]$$

$$\Rightarrow C_3 = 0$$

Updated solution

$$u(x,y) = C_2 \sin\frac{n\pi x}{10} \times C_4 e^{\frac{-n\pi y}{10}} \quad \text{— (10)}$$

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$$u(x,y) = c_2 c_4 \sin\frac{n\pi x}{10} \times e^{-\frac{n\pi y}{10}} \quad \text{— (11)}$$

Most general solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \times e^{-\frac{n\pi y}{10}}$$

— (12)

Apply BC-4

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Apply Half Range sine series

$$b_n = \frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{10}\int_0^{10} u(x,0)\sin\frac{n\pi x}{10}\,dx$$

$$b_n = \frac{1}{5}\left[\int_0^5 u(x,0)\sin\frac{n\pi x}{10}dx + \int_5^{10} u(x,0)\sin\frac{n\pi x}{10}dx\right]$$

$$b_n = \frac{1}{5}\int_0^5 20x\sin\frac{n\pi x}{10}dx + \frac{1}{5}\int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}dx$$

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$$b_n = \frac{20}{5}\int_0^5 x \sin\frac{n\pi x}{10}\,dx + \frac{20}{5}\int_5^{10} (10-x)\sin\frac{n\pi x}{10}\,dx$$

$$b_n = 4\left[x\frac{\left(-\cos\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} - 1x\frac{\left(-\sin\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_0^5 + 4\left[(10-x)\frac{\left(-\cos\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} - (-1)\frac{\left(-\sin\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_5^{10}$$

$$b_n = 4\left[\frac{-5\cos\frac{n\pi}{2}}{\frac{n\pi}{10}} + \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{10}\right)^2}\right] + 4\left[0+0-\left(\frac{-5\cos\frac{n\pi}{2}}{\frac{n\pi}{10}} - \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]$$

$$b_n = 4\left(-\frac{50}{n\pi}\cos n\pi + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + 4\left(\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

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$$b_n = \frac{-200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{\pi^2 n^2}\sin\frac{n\pi}{2} + \frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

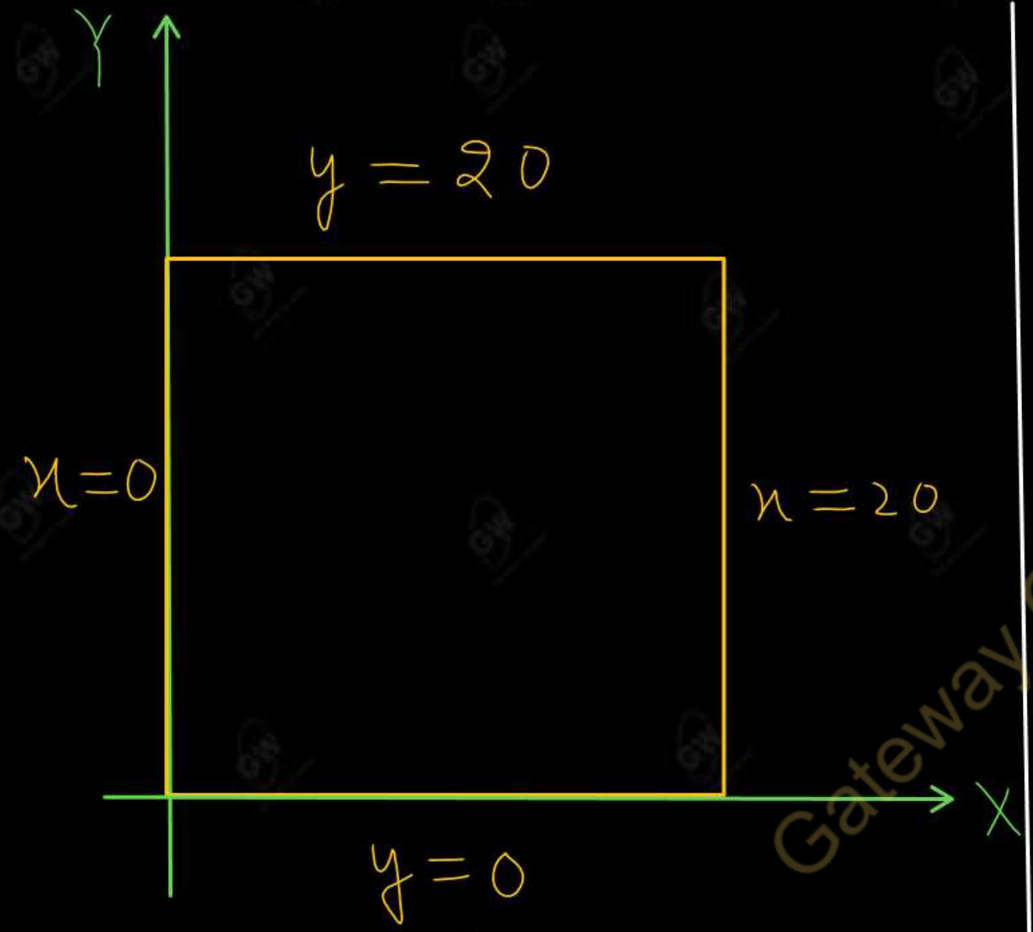
Put b_n in (12)

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

$$u(x,y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

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Q.2:- A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while other three edges are kept 0°C . Find the steady state temperature in the plate. *(A.K.T.U. 2014, 2017)*



Let $u(x, y)$ be the steady state temp given by Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

Under the Boundary conditions

$u(0, y) = 0$ BC-1

$u(x, 0) = 0$ BC-2

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$u(20, y) = 0$ BC-3

$u(x, 20) = x(20 - x)$ BC-4

Solution of eqn (1)

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (1)}$$

Apply BC-1

$$0 = C_1 (C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow C_4 = 0$$

Updated solution

$$u(x, y) = C_2 \sin px (C_3 e^{py} + C_4 e^{-py}) \quad \text{— (2)}$$

Apply BC-2

$$0 = C_2 \sin px (C_3 + C_4)$$

$$\Rightarrow C_3 + C_4 = 0$$

$$C_4 = -C_3$$

Updated solution

$$u(x, y) = C_2 \sin px (C_3 e^{py} - C_3 e^{-py})$$

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$$u(x,y) = c_2 c_3 \sin px \left(e^{py} - e^{-py}\right) \quad \text{— (9)}$$

Apply BC-3

$$0 = c_2 c_3 \sin 20p \left(e^{py} - e^{-py}\right)$$

$$\Rightarrow \sin 20p = 0$$

$$\sin 20p = \sin n\pi$$

$$20p = n\pi$$

$$p = \frac{n\pi}{20}$$

Updated solution

$$u(x,y) = c_2 c_3 \sin \frac{n\pi}{20} x \left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right) \quad \text{— (10)}$$

Most General solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right) \quad \text{— (11)}$$

Apply BC-4

$$x(20-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \left(e^{n\pi} - e^{-n\pi}\right)$$

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$$(20x - x^2) = \sum_{n=1}^{\infty} b_n' \sin\frac{n\pi x}{20}$$

Apply Half range sine-series

$$b_n' = \frac{2}{l}\int_0^l u(x,20)\sin\frac{n\pi x}{l}\,dx$$

$$b_n' = \frac{2}{20}\int_0^{20} (20x - x^2)\sin\frac{n\pi x}{20}\,dx$$

I II

$$b_n' = \frac{1}{10}\left[(20x - x^2)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - (20-2x)\left(\frac{-\sin\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^2}\right) + (-2)\left(\frac{+\cos\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^3}\right)\right]_0^{20}$$

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$$b_n' = \frac{1}{10}\left[0 + 0 - \frac{2\cos n\pi}{\left(\frac{n\pi}{20}\right)^3} - \left(0 + 0 - \frac{2}{\left(\frac{n\pi}{20}\right)^3}\right)\right]$$

$$b_n' = \frac{1}{10}\left[-\frac{2\times 8000}{n^3\pi^3}\cos n\pi + \frac{2\times 8000}{n^3\pi^3}\right]$$

$$b_n' = \frac{2\times 8000}{10\times n^3\pi^3}\left[1 - \cos n\pi\right]$$

$$b_n' = \frac{1600}{n^3\pi^3}\left[1 - (-1)^n\right]$$

When n is even

$$b_n' = 0$$

When n is odd

$$b_n' = \frac{3200}{n^3\pi^3}$$

$$b_n\left(e^{n\pi} - e^{-n\pi}\right) = \frac{3200}{n^3\pi^3}$$

$$2b_n\left(\frac{e^{n\pi} - e^{-n\pi}}{2}\right) = \frac{3200}{n^3\pi^3}$$

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$$2 b_n \sinh n\pi = \frac{3200}{n^3\pi^3}$$

$$b_n = \frac{1 \times 3200}{2\sinh n\pi \times n^3\pi^3}$$

Put b_n in (11)

$$u(x,y) = \sum_{n=1}^{\infty} \frac{3200}{2\sinh n\pi \times n^3\pi^3} \sin\frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{20} \times 2}{2n^3\sinh n\pi}\left(\frac{e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}}{2}\right)$$

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{20}}{n^3}\frac{\sinh\left(\frac{n\pi y}{20}\right)}{\sinh n\pi}$$

Where n is odd

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\frac{(2n-1)\pi x}{20} \times \sinh\frac{(2n-1)\pi y}{20}}{(2n-1)^3\sinh(2n-1)\pi}$$

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Engineering Mathematics by Gulshan Sir

DPP- 10

Topic : Two Dimensional Heat Equation (Laplace Equation)

Part-2

Q.1. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle in the xy – plane with $u(x, 0) = 0$, $u(x, b) = 0$, $u(0,y) = 0$ and $u(a, y) = f(y)$ parallel to y-axis. (U.K.T.U. 2016)

Q.2 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0, y) = 0$, $u(a, y) = 0$, $u(x, 0) = 0$ and $u(x, b) = x$. *[U.P.T.U. 2015]*

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ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Complex Fourier transform***
- PYQ
- DPP

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Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

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Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or $f(p)$ and defined as

$$F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx$$

F(s)

$i = \sqrt{-1}$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipx}dp$$

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Some Important Points and Formulae

1. Even and Odd Function :

(i) Even Function : A function is said be even if

$$f(-x) = f(x)$$

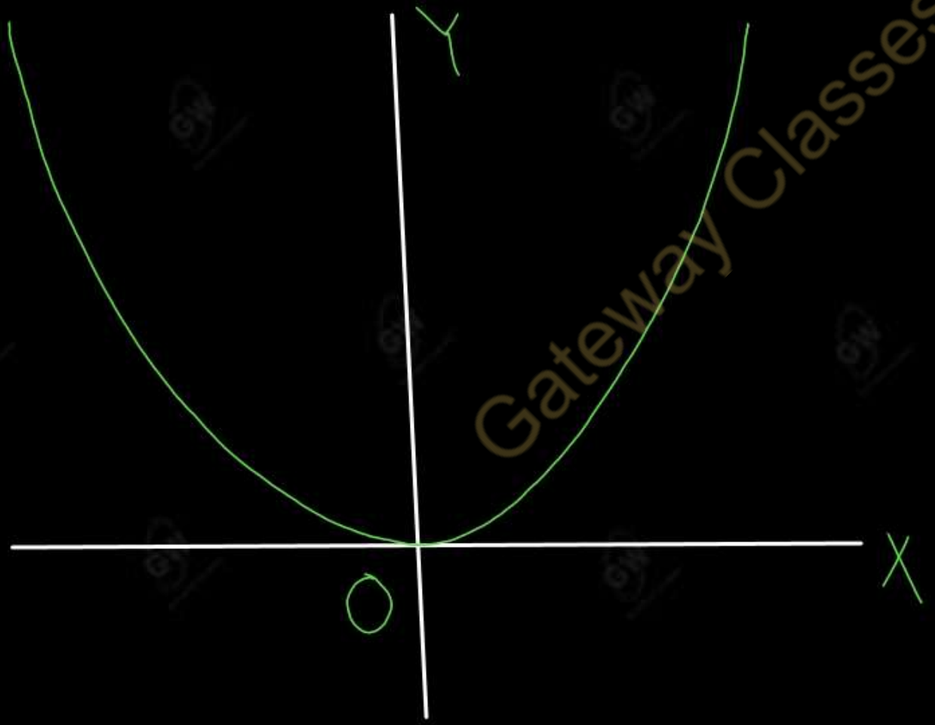
Note: Graph of even function is symmetrical about Y-axis

Example

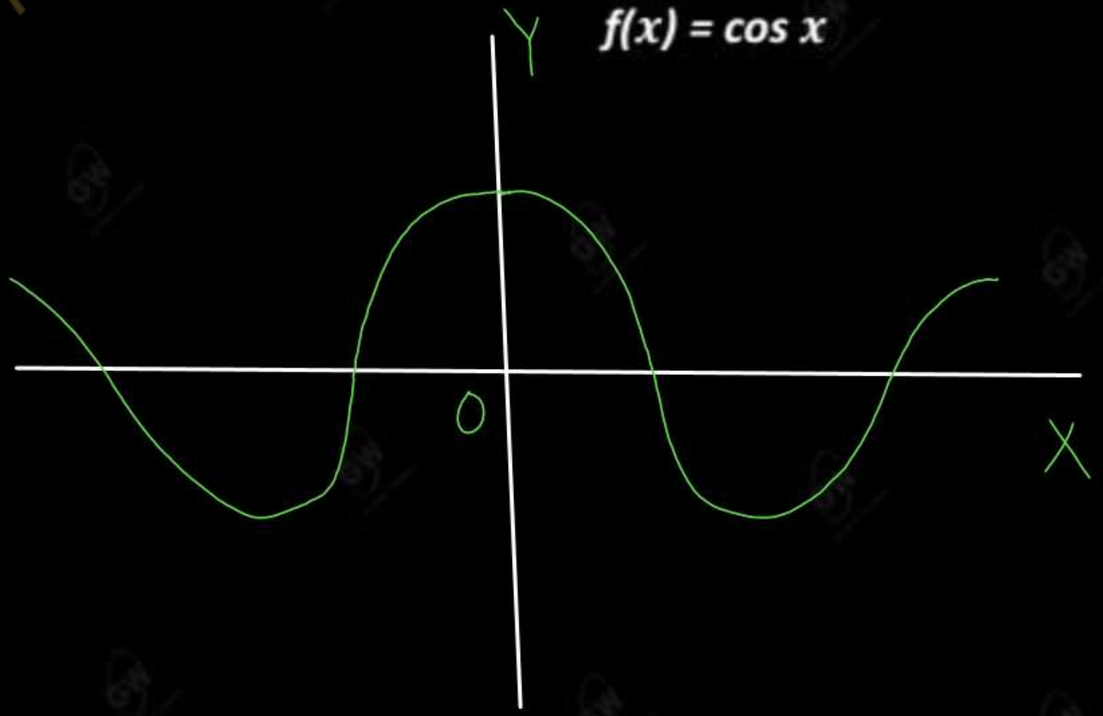
(1) $f(x) = \cos x$

(2) $f(x) = x^2$

$f(x) = x^2$



$f(x) = \cos x$



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(ii) Odd Function : A function is said be Odd if

$$f(-x) = -f(x)$$

Example

(a) $f(x) = \sin x$

(b) $f(x) = \sin^3 x$

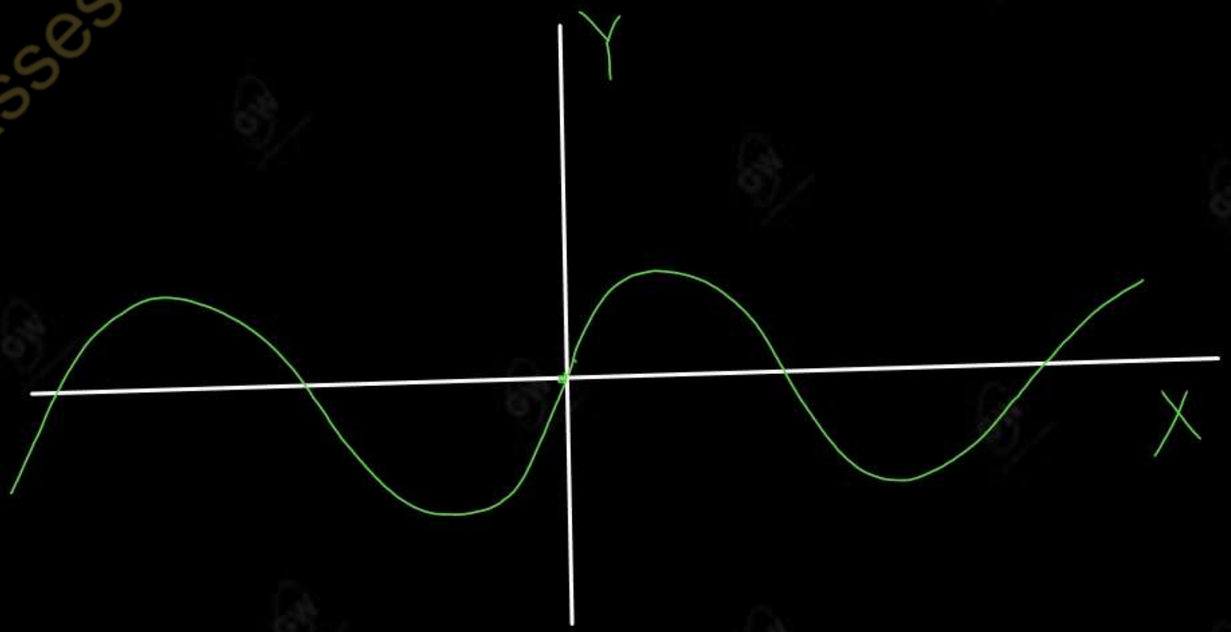
(c) $f(x) = x$

Note: The graph $f(x)$ of an odd function is symmetrical about origin and in opposite quadrant.

$f(x) = x$



$f(x) = \sin x$



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Also Remember

(i) Even function × Even function = Even function

(ii) Odd function×Odd function = Even function

(iii) Even function × Odd function = Odd function

(iv) Odd function×Even function = Odd function

② An important Property of Definite integral

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \qquad \text{If } f(x) \text{ is an even function}$$

$$= 0 \qquad \text{If } f(x) \text{ is an odd function}$$

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③ Integration By-Parts

$$\int_{I\ II} uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

④ $$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

⑤ $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

⑥ $$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Gateway Classes : 7455 961284

NOTE:

① $|x| \leq a$

$-a \leq x \leq a$

② $|x| < a$

$-a < x < a$

③ $|x| \geq a$

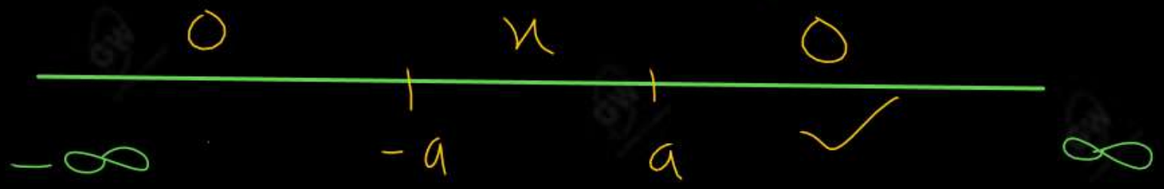
$x \geq a$

$x \leq -a$

④ $|x| > a$

$x > a$

$x < -a$



Gateway Classes : 7455 9612 84

Q.1. Find the Fourier Transform of $F(x) = \begin{cases} x, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$

$$F(x) = \begin{cases} x & -a < x < a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{-a} F(x) e^{ipx} dx + \int_{-a}^{a} F(x) e^{ipx} dx + \int_{a}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = 0 + \int_{-a}^{a} x\, e^{ipx} dx + 0$$

$$f(p) = \int_{-a}^{a} x\, e^{ipx} dx$$

I II

$$f(p) = \left[x\left(\frac{e^{ipx}}{ip}\right) - 1\left(\frac{e^{ipx}}{i^2 p^2}\right) \right]_{-a}^{a}$$

Gateway Classes : 7455 9612 84

$$f(p) = \frac{a e^{ipa}}{ip} - \frac{e^{ipa}}{i^2 p^2} - \left(\frac{-a e^{-ipa}}{ip} - \frac{e^{-ipa}}{i^2 p^2} \right)$$

$$f(p) = \frac{a e^{ipa}}{ip} + \frac{e^{ipa}}{p^2} + \frac{a e^{-ipa}}{ip} - \frac{e^{-ipa}}{p^2}$$

$$f(p) = \frac{a}{ip}\left(e^{ipa} + e^{-ipa} \right) + \frac{1}{p^2}\left(e^{ipa} - e^{-ipa} \right)$$

$$f(p) = \frac{2a}{ip}\left(\frac{e^{ipa} + e^{-ipa}}{2} \right) + \frac{2i}{p^2}\left(\frac{e^{ipa} - e^{-ipa}}{2i} \right)$$

$$f(p) = \frac{2a}{ip}\cos ap + \frac{2i}{p^2}\sin ap$$

$$f(p) = \frac{2i}{p^2}\left(\frac{ap\cos ap}{i^2} + \sin ap \right)$$

$$\boxed{f(p) = \frac{2i}{p^2}\left(\sin ap - ap\cos ap \right)}$$

Gateway Classes 7455 9612 84

Q.2 . Final the complex Fourier transform of $f(x) = e^{-ax^2}$, $(a > 0)$. Hence find the Fourier transform of

(i) $F(x) = e^{-x^2}$

(ii) $F(x) = e^{-x^2/2}$ **(AKTU-2017)**

$$F(x) = e^{-ax^2}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-ax^2} \times e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-ax^2 + ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x^2 - \frac{ipx}{a}\right)}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[x^2 - \frac{ip}{a}x + \left(\frac{ip}{2a}\right)^2 - \left(\frac{ip}{2a}\right)^2\right]}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[\left(x - \frac{ip}{2a}\right)^2 - \frac{i^2p^2}{4a^2}\right]}$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[\left(x - \frac{ip}{2a}\right)^2 + \frac{p^2}{4a^2}\right]}\, dx$$

Gateway Classes : 7455 9612 84

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{ip}{2a}\right)^2 - \frac{ap^2}{4a^2}} dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{ip}{2a}\right)^2} \times e^{\frac{-p^2}{4a}} dx$$

$$f(p) = e^{-p^2/4a} \int_{-\infty}^{\infty} e^{-\left[\sqrt{a}\left(x - \frac{ip}{2a}\right)\right]^2} dx$$

Put $\sqrt{a}\left(x - \frac{ip}{2a}\right) = t$

$$\sqrt{a}\, dx = dt$$

$$dx = \frac{dt}{\sqrt{a}}$$

$$f(p) = e^{\frac{-p^2}{4a}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{a}}$$

$$f(p) = \frac{e^{\frac{-p^2}{4a}}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$\boxed{f(p) = \frac{e^{\frac{-p^2}{4a}} \times \sqrt{\pi}}{\sqrt{a}}} \quad \text{---(1)}$$

GW GATEWAY CLASSES
Gateway Classes : 7455 9612 84

(i) $F(x) = e^{-x^2}$

Here $a = 1$

Put $a =$ in (1)

$$f(p) = e^{-\frac{p^2}{4}} \times \sqrt{\pi}$$

(ii) $F(x) = e^{-\frac{x^2}{2}}$

Here $a = \frac{1}{2}$

Put $a = \frac{1}{2}$ in (1)

$$f(p) = e^{-\frac{p^2}{2}} \times \sqrt{\pi} \times \sqrt{2}$$

$$f(p) = e^{-\frac{p^2}{2}} \times \sqrt{2\pi}$$

Gateway Classes : 7455 961284

Q.3 . Find the Fourier transform of

$$F(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$$

(UPTU-2016, 2014 & 2010)

Hence evaluate (i) $\int_{-\infty}^{\infty} \left(\frac{\sin as \cos xs}{s}\right) ds.$ (ii) $\int_{0}^{\infty} \left(\frac{\sin s}{s}\right) ds.$

$$F(x) = \begin{cases} 1 & -a < x < a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = \int_{-\infty}^{-a} F(x) e^{ipx} dx + \int_{-a}^{a} F(x) e^{ipx} dx + \int_{a}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = 0 + \int_{-a}^{a} 1 \times e^{ipx} dx + 0$$

$$f(p) = \int_{-a}^{a} e^{ipx} dx$$

Gateway Classes : 7455961234

$$f(p) = \left(\frac{e^{ipx}}{ip}\right)_{-a}^{a}$$

$$f(p) = \frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip}$$

$$f(p) = \frac{1}{ip}\left(e^{ipa} - e^{-ipa}\right)$$

$$f(p) = \frac{2i}{ip}\left(\frac{e^{ipa} - e^{-ipa}}{2i}\right)$$

$$\boxed{f(p) = \frac{2}{p}\sin ap}$$

(i) By Inverse Fourier transform

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin ap}{p}\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{p}\sin ap\, e^{-ipx}\, dp$$

Gateway Classes : 7455 961284

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{p}\sin ap(\cos px - i\sin px)\,dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin ap \cos px}{p} - i\frac{\sin ap \sin px}{p}\right)dp$$

even odd

By property of Definite integral

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p}\,dp$$

Put $p = s$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\,ds$$

$$\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\,ds = \pi F(x)$$

$$\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\,ds = \pi F(x)$$

Gateway Classes: 7455 9612 84

$$\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \pi \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases}$$

(11) Put $a = 1$ and $x = 0$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi F(0)$$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi \cdot 1$$

even

By the property of Definite integral

$$2\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

$$\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

Gateway Classes : 7455 9612 84

Q.4 Find the Fourier transform of $\begin{cases} 1-x^2, & if\ |x|<1 \\ 0, & if\ |x|>1 \end{cases}$ **and use it to Evaluate** $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx.$ **(UPTU-2015)**

$$F(x)=\begin{cases} 1-x^2 & -1<x<1 \\ 0 & x>1 \text{ or } x<-1 \end{cases}$$

By Fourier transform

$$f(p)=\int_{-\infty}^{\infty} F(x)\,e^{ipx}\,dx$$

$$f(p)=\int_{-\infty}^{-1} F(x)e^{ipx}dx+\int_{-1}^{1} F(x)e^{ipx}dx+\int_{1}^{\infty} F(x)e^{ipx}dx$$

$$f(p)=0+\int_{-1}^{1}(1-x^2)e^{ipx}dx+0$$

$$f(p)=\int_{-1}^{1}(1-x^2)e^{ipx}dx$$

I II

$$f(p)=\left[(1-x^2)\left(\frac{e^{ipx}}{ip}\right)-(-2x)\left(\frac{e^{ipx}}{i^2p^2}\right)+(-2)\left(\frac{e^{ipx}}{i^3p^3}\right)\right]_{-1}^{1}$$

Gateway Classes : 7455 961284

$$f(p) = \left[(1-n^2) \frac{e^{ipn}}{ip} - \frac{2ne^{ipn}}{p^2} + \frac{2e^{ipn}}{ip^3} \right]_{-1}^{1}$$

$$f(p) = \left[0 - \frac{2e^{ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \left(0 + \frac{2e^{-ip}}{p^2} + \frac{2e^{-ip}}{ip^3} \right) \right]$$

$$f(p) = -\frac{2e^{ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \frac{2e^{-ip}}{p^2} - \frac{2e^{-ip}}{ip^3}$$

$$f(p) = -\frac{2}{p^2}\left(e^{ip} + e^{-ip} \right) + \frac{2}{ip^3}\left(e^{-ip} - e^{-ip} \right)$$

Gateway Classes : 7455 9612 84

$$f(p) = -\frac{4}{p^2}\left(\frac{e^{ip}+e^{-ip}}{2}\right) + \frac{4}{p^3}\left(\frac{e^{ip}-e^{-ip}}{2i}\right)$$

$$f(p) = -\frac{4}{p^2}\cos p + \frac{4}{p^3}\sin p$$

$$\boxed{f(p) = \frac{4}{p^3}\left(\sin p - p\cos p\right)}$$

By inverse Fourier transform

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^3}\left(\sin p - p\cos p\right) e^{-ipx}\, dp$$

Gateway Classes 7455 961284

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4^2}{p^3}(\sin p - p\cos p)(\cos px - i\sin px)\,dp$$

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\underbrace{\frac{\sin p\cos px}{p^3}}_{\text{even}} \underbrace{- i\frac{\sin p\sin px}{p^3}}_{\text{odd}} \underbrace{- \frac{p\cos p\cos px}{p^3}}_{\text{even}} + \underbrace{i\frac{p\cos p\sin px}{p^3}}_{\text{odd}}\right)dp$$

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p\cos px}{p^3} - \frac{p\cos p\cos px}{p^3}\right)dp$$

Gateway Classes 7455 9612 84

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos px\,dx$$

Put $x = \frac{1}{2}$

$$F\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos\frac{p}{2}\,dp$$

Replace p by x

$$F\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx = \frac{\pi}{2}F\left(\frac{1}{2}\right)$$

$$\int_{-\infty}^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{\pi}{2}F\left(\frac{1}{2}\right) \quad \text{— (1)}$$

$$F(x) = 1 - x^2$$

$$F\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2$$

$$F\left(\frac{1}{2}\right) = \frac{3}{4}$$

Gateway Classes : 7455 961284

$$\int_{-\infty}^{\infty} \left(\frac{x\cos x - \sin x}{x^3} \right) \cos\frac{x}{2}\, dx = -\frac{3\pi}{8}$$

Gateway Classes : 7455 96 12 84

DPP- 11

Topic : *Complex Fourier transform*

Q.1. *Find the Fourier Transform of* $F(x) = \begin{cases} x^2, & if\ |x| < a \\ 0, & if\ |x| > a \end{cases}$

Q.2. *Find the Fourier Transform of* $f(x) = e^{-|x|}$

Q.3. *Find the Fourier transform of* $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. *hence avaluate* $\int_0^\infty \frac{\sin x}{x} dx$. *(GBTU-2011)*

Q. 4. *Find the Fourier transform of Block function* $f(t)$ *of height 1 and duration a defined by* *(AKTU-2016)*

$$f(t) = \begin{cases} 1, & for\ |t| \le \frac{a}{2} \\ 0, & otherwise \end{cases}$$

Gateway Classes : 7455 961 284

ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Fourier sine transform***
- PYQ
- DPP

Gateway Classes : 7455 9012 84

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Gateway Classes, 7455 9612 84

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or f(p) and defined as

OR
$F(s)$

$$F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx$$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipx}dp$$

Gateway Classes : 7455 96 12 84

Fourier sine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$ or $f_s(p)$ and defined as

$$F_s\{F(x)\} = f_s(p) = \int_0^\infty F(x) \sin px\, dx$$

Where p is a parameter and $p > 0$

Inverse Fourier sine transform

If $F_s\{F(x)\} = f_s(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier sine transform and defined as

$$F(x) = F_s^{-1}\{f_s(p)\} = \frac{2}{\pi} \int_0^\infty f_s(p) \sin px\, dp$$

Gateway Classes : 7455 96 19 84

Some Important Points and Formulae

GW GATEWAY CLASSES

Two Important results

(i) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + c$

(ii) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + c$

Trigonometric Identities

(1) $2 \sin A \cos A = \sin 2A$

(2) $2 \sin A \cos B = \sin (A+B) + \sin (A - B)$

(3) $2 \cos A \sin B = \sin (A+B) - \sin (A - B)$

(4) $2 \cos A \cos B = \cos (A+B) + \cos (A - B)$

(5) $2 \sin A \sin B = \cos (A-B) - \cos (A + B)$

Gateway Classes : 7455 9612 84

Q.1. *Find the Fourier sine transform of* $F(x) = e^{-3x} + e^{-4x}$

$$F(x) = e^{-3x} + e^{-4x}$$

By Fourier sine transform

$$f_s(p) = \int_0^\infty F(x) \sin px \, dx$$

$$f_s(p) = \int_0^\infty \left(e^{-3x} + e^{-4x}\right) \sin px \, dx$$

$$f_s(p) = \int_0^\infty e^{-3x} \sin px \, dx + \int_0^\infty e^{-4x} \sin px \, dx$$

$$f_s(p) = \left[\frac{e^{-3x}}{(-3)^2+p^2}(-3\sin px - p\cos px)\right]_0^\infty$$

$$+ \left[\frac{e^{-4x}}{(-4)^2+p^2}(-4\sin px - p\cos px)\right]_0^\infty$$

$$f(p) = \left[0 - \frac{1}{9+p^2}(0-p)\right] +$$

$$\left[0 - \frac{1}{16+p^2}(0-p)\right]$$

Gateway Classes : 7455 9612 84

$$f_s(p) = \left[\frac{-1}{9+p^2}(-p)\right] + \left[\frac{-1}{16+p^2}(-p)\right]$$

$$\boxed{f_s(p) = \frac{p}{p^2+9} + \frac{p}{p^2+16}}$$

Gateway Classes : 7455 9612 84

Q.2 . Find the Fourier sine transform of $e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$.

$$F(x) = e^{-|x|}$$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

For Fourier sine transform

$$x > 0$$

$$\Rightarrow |x| = x$$

$$\Rightarrow F(x) = e^{-x}$$

$$f_s(p) = \int_0^\infty F(x) \sin px\, dx$$

$$= \int_0^\infty e^{-x} \sin px\, dx$$

$$= \left[\frac{e^{-x}}{(-1)^2 + p^2} \left(-\sin px - p \cos px \right) \right]_0^\infty$$

$$= \left[0 - \left(\frac{1}{1+p^2} (-p) \right) \right]$$

Gateway Classes : 7455 9612 84

$$f_s(p) = -\left(\frac{-p}{1+p^2}\right)$$

$$\boxed{f_s(p) = \frac{p}{p^2+1}}$$

By Inverse Fourier sine Transform

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(p)\sin px\,dp$$

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2+1}\sin px\,dp$$

Put $x = m$

$$F(m) = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2+1}\sin pm\,dp$$

$$\int_0^\infty \frac{p}{p^2+1}\sin pm\,dp = \frac{\pi}{2}F(m)$$

Replace p by x

$$\boxed{\int_0^\infty \frac{x}{x^2+1}\sin mx\,dx = \frac{\pi}{2}e^{-m}}$$

Gateway Classes : 7455 9812 84

Q.3. Find Fourier sine transform of $\frac{e^{-ax}}{x}$, $a > 0$. Hence find the Fourier sine transform of $\frac{1}{x}$ (GBTU-2011; UPTU-2015)

$$F(x) = \frac{e^{-ax}}{x}$$

By Fourier sine transform

$$f_s(p) = \int_0^\infty F(x) \sin px \, dx$$

$$f_s(p) = \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx$$

Let $f_s(p) = I$

$$I = \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx \quad \text{——— (1)}$$

differentiate w.r.t p

$$\frac{dI}{dp} = \int_0^\infty \frac{e^{-ax}}{x} \frac{d}{dp}(\sin px) \, dx$$

$$\frac{dI}{dp} = \int_0^\infty \frac{e^{-ax}}{x} \times x \cos px \, dx$$

Gateway Classes: 7455 9612 84

$$\frac{dI}{dp} = \int_0^\infty e^{-ax} \cos px \, dx$$

$$\frac{dI}{dp} = \left[\frac{e^{-ax}}{(-a)^2 + p^2} \left(-a\cos px + p \sin px\right) \right]_0^\infty$$

$$\frac{dI}{dp} = \left[0 - \left\{ \frac{1}{a^2+p^2} (-a) \right\} \right]$$

$$\frac{dI}{dp} = \frac{a}{a^2+p^2}$$

$$dI = \frac{a}{a^2+p^2} dp$$

Integrate w.r.t p

$$\int dI = \int \frac{a}{a^2+p^2} dp$$

$$I = \tan^{-1}\left(\frac{p}{a}\right) + c \quad -(2)$$

$$f_s(p) = \tan^{-1}\left(\frac{p}{a}\right) + c$$

Let initially $p = 0$

Gateway Classes : 7455 961284

Put $p = 0$ in (1)

$$I = 0$$

Put $p = 0$ and $I = 0$ in (2)

$$0 = 0 + c$$

$$c = 0$$

$$\boxed{f_s(p) = \tan^{-1}\left(\frac{p}{a}\right)}$$

$$f_s(p) = \tan^{-1}\left(\frac{p}{a}\right)$$

$$\int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx = \tan^{-1}\left(\frac{p}{a}\right)$$

Put $a = 0$

$$\int_0^\infty \frac{1}{x} \sin px \, dx = \tan^{-1}\left(\frac{p}{0}\right) = \tan^{-1}(\infty)$$

$$\boxed{\int_0^\infty \frac{1}{x} \sin px \, dx = \frac{\pi}{2}}$$

Gateway Classes : 7455 961284

Q.4. *Find $F(x)$ if its Fourier sine transform is e^{-ap}*

$$f_s(p) = e^{-ap}$$

By inverse Fourier sine transform

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(p)\sin px\,dp$$

$$= \frac{2}{\pi}\int_0^\infty e^{-ap}\sin px\,dp$$

$$= \frac{2}{\pi}\left[\frac{e^{-ap}}{(-a)^2+x^2}\left(-a\sin px - x\cos px\right)\right]_0^\infty$$

$$F(x) = \frac{2}{\pi}\left[0 - \left\{\frac{1}{a^2+x^2}(-x)\right\}\right]$$

$$F(x) = \frac{2}{\pi}\left(\frac{x}{a^2+x^2}\right)$$

$$\boxed{F(x) = \frac{2x}{\pi(a^2+x^2)}}$$

Gateway Classes : 7455961284

DPP- 12

Topic : *Fourier sine transform*

Q.1. *Find Fourier sine transform of* $\frac{1}{x}$

Q. 2. *Find the Fourier sine transform of Block function* $f(t)$ *of height 1 and duration a defined by*

$$f(t) = \begin{cases} sinx, & 0 < x < a \\ 0, & x > a \end{cases}$$

Q.3. *Find* $F_s^{-1}\left(\frac{e^{-ap}}{p}\right)$ *and hence evaluate* $F_s^{-1}\left(\frac{1}{p}\right)$. *(AKTU-2016)*

Gateway Classes : 7455 9612 84

ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Fourier cosine transform***
- PYQ
- DPP

Gateway Classes : 7455 9012 84

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Gateway Classes, 7455 9612 84

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or f(p) and defined as

$$F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx$$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipx}dx$$

Gateway Classes : 7455 96 12 84

Fourier sine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$ or $f_s(p)$ and defined as

$$F\{F(x)\} = f_s(p) = \int_0^\infty F(x) sinpx dx$$

Where p is a parameter and $p > 0$

Inverse Fourier sine transform

If $F_s\{F(x)\} = f_s(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier sine transform and defined as

$$F_s^{-1}\{f_s(p)\} = F(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} f_s(p)\, sinpx dp$$

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Fourier cosine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier cosine transform of $F(x)$ is denoted by $F_c\{F(x)\}$ or $f_c(p)$ and defined as

$$F_c\{F(x)\} = f_c(p) = \int_0^\infty F(x)\cos px\,dx$$

Where p is a parameter and $p > 0$

Inverse Fourier cosine transform

If $F_c\{F(x)\} = f_c(p)$ is the Fourier cosine transform of $F(x)$, then $F(x)$ is called inverse Fourier cosine transform and defined as

$$F_c^{-1}\{f_c(p)\} = F(x) = \frac{2}{\pi}\int_0^\infty f_c(p)\cos px\,dp$$

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Some Important Points and Formulae

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Two Important results

(i) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + c$

(ii) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + c$

Trigonometric Identities

(1) $2 \sin A \cos A = \sin 2A$

(2) $2 \sin A \cos B = \sin (A+B) + \sin (A - B)$

(3) $2 \cos A \sin B = \sin (A+B) - \sin (A - B)$

(4) $2 \cos A \cos B = \cos (A+B) + \cos (A - B)$

(5) $2 \sin A \sin B = \cos (A-B) - \cos (A + B)$

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Q.1. *Find the Fourier cosine transform of* $F(x) = e^{-2x} + 4e^{-3x}$

$$F(x) = e^{-2x} + 4e^{-3x}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x)\cos px\,dx$$

$$f_c(p) = \int_0^\infty (e^{-2x} + 4e^{-3x})\cos px\,dx$$

$$f_c(p) = \int_0^\infty e^{-2x}\cos px\,dx + 4\int_0^\infty e^{-3x}\cos px\,dx$$

$$f_c(p) = \left[\frac{e^{-2x}}{(-2)^2 + p^2}(-2\cos px + p\sin px)\right]_0^\infty$$

$$+ 4\left[\frac{e^{-3x}}{(-3)^2 + p^2}(-3\cos px + p\sin px)\right]_0^\infty$$

$$f_c(p) = \left[0 - \left\{\frac{1}{4+p^2}(-2)\right\}\right]$$

$$+ 4\left[0 - \left\{\frac{1}{9+p^2}(-3)\right\}\right]$$

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$$f_c(p) = \frac{2}{p^2+4} + \frac{12}{p^2+9}$$

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Q.2 . Find Fourier cosine transform of $F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

$$F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^a F(x) \cos px \, dx + \int_a^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^a \cos x \cos px \, dx + 0$$

$$f_c(p) = \frac{1}{2} \int_0^a 2 \cos x \cos px \, dx$$

$$f_c(p) = \frac{1}{2} \int_0^a \{\cos(x+px) + \cos(x-px)\} dx$$

$$f_c(p) = \frac{1}{2} \int_0^a \{\cos(1+p)x + \cos(1-p)x\} dx$$

$$f_c(c) = \frac{1}{2} \left[\frac{\sin(1+p)x}{1+p} + \frac{\sin(1-p)x}{1-p} \right]_0^a$$

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$$f_c(p) = \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p} - (0+0)\right]$$

$$\boxed{f_c(p) = \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p}\right]}$$

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Q.3 . Obtain Fourier cosine transform of $F(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

$$F(x) = \begin{cases} x & ; \quad 0 < x < 1 \\ 2 - x & ; \quad 1 < x < 2 \\ 0 & ; \quad x > 2 \end{cases}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^1 F(x) \cos px \, dx + \int_1^2 F(x) \cos px \, dx + \int_2^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^1 x \cos px \, dx + \int_1^2 (2 - x) \cos px \, dx + 0$$

$$f_c(p) = \left[x \left(\frac{\sin px}{p} \right) - 1 \left(\frac{-\cos px}{p^2} \right) \right]_0^1$$

$$+ \left[(2 - x) \left(\frac{\sin px}{p} \right) - (-1) \left(\frac{-\cos px}{p^2} \right) \right]_1^2$$

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$$f_c(p) = \left[\frac{x \sin px}{p} + \frac{\cos px}{p^2}\right]_0^1 + \left[(2-x)\frac{\sin px}{p} - \frac{\cos px}{p^2}\right]_1^2$$

$$f_c(p) = \frac{\sin p}{p} + \frac{\cos p}{p^2} - \left(0 + \frac{1}{p^2}\right) + 0 - \frac{\cos 2p}{p^2} - \left(\frac{\sin p}{p} - \frac{\cos p}{p^2}\right)$$

$$f_c(p) = \frac{\sin p}{p} + \frac{\cos p}{p^2} - \frac{1}{p^2} - \frac{\cos 2p}{p^2} - \frac{\sin p}{p} + \frac{\cos p}{p^2}$$

$$\boxed{f_c(p) = \frac{2\cos p}{p^2} - \frac{\cos 2p}{p^2} - \frac{1}{p^2}}$$

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$$f_c(p) = \frac{2\cos p}{p^2} - \left(\frac{2\cos^2 p - 1}{p^2}\right) - \frac{1}{p^2} \qquad \{\because \cos 2\theta = 2\cos^2\theta - 1\}$$

$$f_c(p) = \frac{2\cos p}{p^2} - \frac{2\cos^2 p}{p^2} + \frac{1}{p^2} - \frac{1}{p^2}$$

$$f_c(p) = \frac{2\cos p - 2\cos^2 p}{p^2}$$

$$f_c(p) = \frac{2\cos p(1-\cos p)}{p^2}$$

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Q.4. Find Fourier cosine transform of $\frac{1}{1+x^2}$ and hence find Fourier sine transform of $\frac{x}{1+x^2}$. (AKTU-2018, 2017)

$$F(x) = \frac{1}{1+x^2}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$$

Let $f_c(p) = I$

$$I = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx \quad \text{——} (1)$$

Differentiate w.r.t p

$$\frac{dI}{dp} = \int_0^\infty \frac{1}{1+x^2} \frac{d}{dp}(\cos px) \, dx$$

$$\frac{dI}{dp} = \int_0^\infty \frac{1}{1+x^2} \times x (-\sin px) \, dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{x \sin px}{1+x^2} \, dx$$

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$$\frac{dI}{dp} = -\int_0^\infty \frac{x^2 \sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{(1+x^2-1)\sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{(1+x^2)\sin px - \sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\left[\int_0^\infty \left\{\frac{(1+x^2)\sin px}{x(1+x^2)} - \frac{\sin px}{x(1+x^2)}\right\}dx\right.$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{\sin px}{x}\,dx + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx \quad \text{---(2)}$$

Differentiate w.r.t p

$$\frac{d^2I}{dp^2} = 0 + \int_0^\infty \frac{x\cos px}{x(1+x^2)}\,dx$$

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$$\frac{d^2I}{dp^2} = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$$

$$\frac{d^2I}{dp^2} = I$$

$$\frac{d^2I}{dp^2} - I = 0$$

solution of above Differential equation is

$$I = c_1 e^{p} + c_2 e^{-p}$$

$$I = c_1 e^{p} + c_2 e^{-p} \quad \text{— (3)}$$

diff w.r.t p

$$\frac{dI}{dp} = c_1 e^{p} - c_2 e^{-p} \quad \text{— (4)}$$

When $p = 0$

From (3)

$$I = c_1 + c_2 \quad \text{— (5)}$$

From (4)

$$\frac{dI}{dp} = c_1 - c_2 \quad \text{— (6)}$$

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From (1)

$$I = \int_0^\infty \frac{dx}{1+x^2}$$

$$I = (\tan^{-1} x)_0^\infty$$

$$I = \tan^{-1}\infty - \tan^{-1} 0$$

$$I = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{2}$$

From (2)

$$\frac{dI}{dp} = -\frac{\pi}{2}$$

Put I and $\frac{dI}{dp}$ in (5) and (6)

$$c_1 + c_2 = \frac{\pi}{2} \quad \text{— (7)}$$

$$c_1 - c_2 = -\frac{\pi}{2} \quad \text{— (8)}$$

Solve (7) and (8)

$$c_1 + c_2 = \frac{\pi}{2}$$
$$c_1 - c_2 = -\frac{\pi}{2}$$

$$2c_1 = 0$$

$$\boxed{c_1 = 0}$$

$$\boxed{c_2 = \frac{\pi}{2}}$$

Put c_1 and c_2 in (3)

$$I = \frac{\pi}{2} e^{-p}$$

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$$\int_0^\infty \frac{1}{1+x^2}\cos px\,dx = \frac{\pi}{2}e^{-p}$$

$$\int_0^\infty \frac{x}{1+x^2}\sin px\,dx = \frac{\pi}{2}e^{-p}$$

diff w r t p

$$\int_0^\infty \frac{1}{1+x^2}\frac{d}{dp}(\cos px)\,dx = \frac{\pi}{2}\frac{d}{dp}(e^{-p})$$

$$\int_0^\infty -\frac{x\sin px}{1+x^2}\,dx = -\frac{\pi}{2}e^{-p}$$

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DPP- 13

Topic : *Fourier cosine transform*

Q.1. *Find Fourier cosine transform* $F(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1 \\ 0, & x > 1 \end{cases}$

Q. 2. *Find the Fourier cosine transform of* e^{-x^2}

Q.3. *Find Fourier cosine transform of* $F(x) = 2e^{-5x} + 2e^{-2x}$

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