Engineering Mathematics

Unit: Partial Differential Equations (PDE)

Lec-4

Today's Target

- Lagrange's Method [Part-1]
- Univ. Questions
- Practice Questions



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Lagrange's Method

First order Quasi-Linear PDE in standard form is called Lagrange's Equation

$$Pp + Qq = R$$

Where P, Q, R are function of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Step to Solve questions

Step-1 Write down the PDE in standard form

Step-2 Write down the Auxiliary Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{— (1)}$$

(I) (II) (III)

Step-3 Solve above auxiliary Equation by

(i) Method of Grouping

(ii) Method of Multipliers

Now we get two independent solutions in the form of

$$u(x, y, z) = c_1 \quad \text{—— (2)}$$

$$v(x, y, z) = c_2 \quad \text{— (3)}$$

Step-4 General Solution

$\phi(u, v) = 0$ **OR** $u = \phi(v)$ **OR** $v = \phi(u)$

Method of Grouping

Type-1

Q.1 Solve : $p + q = z$

$p + q = z$

Here $P = 1$

$Q = 1$

$R = z$

Aux. Eq^n

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z}$$

I II III

III ⟵ (1)

(I, II)

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrate both side

$$\int dx = \int dy$$

$$x = y + c_1$$

$$x - y = c_1 \quad \text{—(2)}$$

(I, III)

$$\frac{dx}{1} = \frac{dz}{z}$$

Integrate both side

$$\int dx = \int \frac{dz}{z}$$

$$x = \log z + c_2$$

$$x - \log z = c_2 \quad \text{⟵ (3)}$$

General solution

$$\phi(x - y, x - \log z) = 0$$

Q.2 Solve : $px + qy = z$

$Px + Qy = z$

Here

$P = x, \quad Q = y, \quad R = z$

Aux. Eqn

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

I II III

(I, II)

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x = \log(y \times c_1)$$

$$x = y \times c_1$$

$$\frac{x}{y} = c_1 \quad — (1)$$

(II, III)

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_2$$

$$\log y = \log(z \times c_2)$$

$$y = z \times c_2$$

$$\frac{y}{z} = c_2 \quad — (2)$$

General solution

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Q.3 Solve the following partial differential equations : $yzp - xzq = xy$

$$yzp - xzq = xy$$

Here

$P = yz$, $Q = -xz$, $R = xy$

Aux. Eqn

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

I II III

(I, II)

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\frac{y^2}{2} + \frac{x^2}{2} = c$$

$$x^2 + y^2 = 2c$$

$$x^2 + y^2 = c_1 \quad \text{— (1)}$$

(II, III)

$$\frac{dy}{-xz} = \frac{dz}{xy}$$

$$\frac{dy}{-z} = \frac{dz}{y}$$

$$y\,dy = -z\,dz$$

$$\int y\,dy = -\int z\,dz$$

$$\frac{y^2}{2} = -\frac{z^2}{2} + c$$

$$y^2 + z^2 = c_2 \quad \text{— (2)}$$

General solution

$$\phi(x^2+y^2,\ y^2+z^2) = 0$$

(I, III)

$$\frac{dx}{yz} = \frac{dz}{xy}$$

$$x\,dx = z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$x^2 - z^2 = c_2$$

Q.4 Solve the following partial differential equations : $p \tan x + q \tan y = \tan z$

$p \tan x + q \tan y = \tan z$

Here

$P = \tan x$, $Q = \tan y$, $R = \tan z$

Aux. Eqn

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

I II III

(I, II)

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\cot x\, dx = \cot y\, dy$$

$$\int \cot x\, dx = \int \cot y\, dy$$

$$\log \sin x = \log \sin y + \log c_1$$

$$\log \sin x = \log \sin y \times c_1$$

$$\sin x = \sin y \times c_1$$

$$\frac{\sin x}{\sin y} = c_1 \quad \text{— (1)}$$

(II, III)

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\cot y\, dy = \cot z\, dz$$

$$\int \cot y\, dy = \int \cot z\, dz$$

$$\log \sin y = \log \sin z + \log c_2$$

$$\log \sin y = \log \sin z \times c_2$$

$$\frac{\sin y}{\sin z} = c_2 \quad \text{— (2)}$$

$$\phi = \left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

Q.5 *Solve the following differential equations :* $\frac{y^2z}{x}p + xzq = y^2$

$$\frac{y^2z}{x}p + xzq = y^2$$

$$\frac{y^2zp + x^2zq}{x} = y^2$$

$$y^2zp + x^2zq = xy^2$$

Here

$P = y^2z$, $Q = x^2z$, $R = xy^2$

Aux. Eqn

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

(I, II)

$$\frac{dx}{y^2\cancel{z}} = \frac{dy}{x^2\cancel{z}}$$

$$x^2dx = y^2dy$$

$$\int x^2dx = \int y^2dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\frac{x^3 - y^3}{3} = c$$

$$x^3 - y^3 = c_1 \quad -(1)$$

(I, III)

$$\frac{dx}{y^{\cancel{2}}z} = \frac{dz}{xy^{\cancel{2}}}$$

$$xdx = zdz$$

$$\int xdx = \int zdz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c$$

$$\frac{x^2 - z^2}{2} = c$$

$$x^2 - z^2 = c_1 \quad -(2)$$

General solution

$$\phi(x^3 - y^3,\ x^2 - z^2) = 0$$

Q.6 Solve the following partial differential equations : $y^2 p - xyq = x(z-2y)$ (Sem. 2014)

$y^2 p - xyq = x(z-2y)$

Here

$P = y^2,\ Q = -xy,\ R = x(z-2y)$

Aux. Eqn

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

I II III

(I, II)

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\frac{x^2+y^2}{2} = c$$

$$x^2 + y^2 = c_1 \quad -(1)$$

II, III

$$\frac{dy}{-xy} = \frac{dz}{+x(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{z-2y}$$

$$(z-2y)\,dy = -y\,dz$$

$$z\,dy - 2y\,dy = -y\,dz$$

$$z\,dy + y\,dz = 2y\,dy$$

$$d(yz) = 2y\,dy$$

$$\int d(yz) = \int 2y\,dy$$

$$yz = \frac{2y^2}{2} + c_2$$

$$yz = y^2 + c_2$$

$$yz - y^2 = c_2$$

$$y^2 - yz = c_2$$

General solution

$$\phi\left(x^2+y^2,\ y^2-yz\right) = 0$$

Type-2

Q.7 Solve : $p+3q=5z+\tan(y-3x)$ **(Sem. 2017)**

$P+3q = 5z + \tan(y-3x)$

Here

$P = 1$

$Q = 3$

$R = 5z + \tan(y-3x)$

Aux. Eqn

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

I II III

(I, II)

$$\frac{dx}{1} = \frac{dy}{3}$$

$$\int dx = \frac{1}{3}\int dy$$

$$x = \frac{y}{3} + c$$

$$3x = y + 3c$$

$$3x - y = 3c$$

$$-1(y-3x) = 3c$$

$$y - 3x = \frac{3c}{-1}$$

$$y - 3x = c_1 \quad -(1)$$

(I, III)

$$\frac{dx}{1} = \frac{dz}{5z+\tan(y-3x)}$$

Put $y-3x = c_1$

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$

$$\int dx = \int \frac{dz}{5z + \tan c_1}$$

Put $5z + \tan c_1 = t$

$$5dz = dt$$

$$dz = \frac{dt}{5}$$

$$\int dx = \frac{1}{5}\int \frac{dt}{t}$$

$$x = \frac{1}{5}\log t + c$$

$$x = \frac{\log t + 5c}{5}$$

$$5x = \log(5z + \tan c_1) + 5c$$

$$5x - \log(5z + \tan(y - 3x)) = c_2$$

$$5x - \log[5z + \tan(y - 3x) = c_2$$

General solution

$$\phi\left[y - 3x, 5x - \log\{5z + \tan(y - 3x)\}\right] = 0$$

Q.8 Solve the following differential equations : $pz - qz = z^2 + (x+y)^2$

$pz - qz = z^2 + (x+y)^2$

Here

$P = z$, $Q = -z$

$R = z^2 + (x+y)^2$

Aux. Eqn

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

I II III

(I, II)

$$\frac{dx}{z} = \frac{dy}{-z}$$

$dy = -dx$

$\int dy = -\int dx$

$y = -x + c_1$

$x + y = c_1$

(I, III)

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$$

Put $x + y = c_1$

$$\frac{dx}{z} = \frac{dz}{z^2 + c_1^2}$$

$$\int dx = \int \frac{z\,dz}{z^2 + c_1^2}$$

Put $z^2 + c_1^2 = t$

$2z\,dz = dt$

$z\,dz = \frac{dt}{2}$

$$\int dx = \frac{1}{2}\int \frac{dt}{t}$$

$$x = \frac{1}{2}\log t + c$$

$$2x = \log(z^2 + c_1^2) + 2c$$

$$2x - \log(z^2 + c_1^2) = c_2$$

$$2x - \log\left(z^2 + (x+y)^2\right) = c_2$$

$$2x - \log(z^2 + x^2 + y^2 + 2xy) = c_2$$

$$\log(z^2 + x^2 + y^2 + 2xy) - 2x = c_2$$

General solution

$$\phi\left[x+y,\ 2x - \log(z^2 + x^2 + y^2 + 2xy)\right]$$

OR

$$\phi\left[x+y,\ \log(z^2 + x^2 + y^2 + 2xy) - 2x\right] = 0$$

Q.9 *Solve the following differential equations :* $zxp + yzq = xy$

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