GATEWAY CLASSES
Empowering learners, transforming futures

DPP-9 Maths - IV Unit - 2

Topic- Two Dimensional Heat Equation (Laplace Equation)

Q-1: Solve temperature distribution in a bar of length π which is perfectly insulated at ends $x=0$ and $x=\pi$ which is governed by partial differential equation $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$, $0<x<\pi$, $0<y<\pi$, which satisfies the conditions:

$u(0,y) = u(\pi,y) = u(x,\pi) = 0$ and $u(x,0) = \sin^2 x$

Sol- Laplace Equation

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \quad -(1)$$

where $u(x,y)$ is steady state temp of plate

Under the Boundry condition;

$u(0,y) = 0$ BC-1

$u(\pi,y) = 0$ BC-2

$u(x,\pi) = 0$ BC-3

$u(x,0) = \sin^2 x$ BC-4

Download Our App from Google PlayStore
Helpline Number : 7455961284
PDF Notes PYQs Solution Recorded Lectures

GATEWAY CLASSES
Empowering learners, transforming futures

Solution of Laplace equation

$$u(x,y) = (C_1 + C_2 x)(C_3 + C_4 y) \quad -(5)$$

$$u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad -(6)$$

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad -(7)$$

Most suitable solution

$$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

Apply BC-1

$$0 = C_1 (C_3 e^{py} + C_4 e^{-py})$$

$$\boxed{C_1 = 0}$$

Updated solution, $u(x,y) = C_2 \sin px (C_3 e^{py} + C_4 e^{-py}) \quad -(9)$

Apply BC-2,

$$0 = C_2 \sin p\pi (C_3 e^{py} + C_4 e^{-py})$$

$$\sin p\pi = 0$$
$$p\pi = n\pi$$
$$\boxed{p = n}$$

Updated solution: $u(x,y) = C_2 \sin nx (C_3 e^{ny} + C_4 e^{-ny}) \quad -(10)$

Download Our App from Google PlayStore
Helpline Number : 7455961284
PDF Notes
PYQs Solution
Recorded Lectures

$$u(x,y) = \sin nx \left(C_2C_3 e^{ny} + C_2C_4 e^{-ny}\right)$$

Let $C_2C_3 = A$ $\qquad$ $C_2C_4 = B$

Apply BC-3 :

$$0 = \sin nx \left(Ae^{n\pi} + Be^{-n\pi}\right)$$

$$Ae^{n\pi} + Be^{-n\pi} = 0$$

$$Ae^{n\pi} = -Be^{-n\pi} = -\frac{1}{2}B_n \text{ (say)}$$

$$\boxed{A = -\frac{B_n e^{-n\pi}}{2}} \qquad \boxed{B = \frac{B_n e^{+n\pi}}{2}}$$

Updated solution :

$$u(x,y) = \frac{1}{2}\sin nx \left(-B_n e^{-n\pi} e^{ny} + B_n e^{+n\pi} e^{-ny}\right)$$

$$u(x,y) = 1 \sin nx\, B_n \left[\frac{e^{n(\pi-y)} - e^{-n(\pi-y)}}{2}\right]$$

$$u(x,y) = \sin nx\, B_n \sinh n(\pi-y)$$

$$\because \boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$$

Most general solution :

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} B_n \sinh n(\pi-y) \sin nx} \quad -(11)$$

Download Our App from Google PlayStore
Helpline Number : 7455961284
PDF Notes
PYQs Solution
Recorded Lectures

GATEWAY CLASSES
Empowering learners, transforming futures

Apply BC-4,

$$\sin^2 x = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin nx$$

$$B_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin^2 x \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (1-\cos 2x)\sin nx\, dx$$

$\cos 2x = 1 - 2\sin^2 x$
$2\sin^2 x = 1 - \cos 2x$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin nx - \frac{2}{2}\cos 2x \sin nx\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin nx - \frac{1}{2}\{\sin(n+2)x + \sin(n-2)x\{\right] dx$$

$\therefore 2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n} + \frac{\cos(n+2)x}{2(n+2)} + \frac{\cos(n-2)x}{2(n-2)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-2}{n} + \frac{1}{n+2} + \frac{1}{n-2}\{(-1)^n - 1\{\right], \text{ when } n \neq 2$$

$$B_n \sinh n\pi = \frac{1}{2\pi}\left[\frac{8}{n(n^2-4)}\{(-1)^n - 1\{\right], \text{ when } n \neq 2$$

Download Our App from Google PlayStore
Helpline Number : 7455961284
PDF Notes
PYQs Solution
Recorded Lectures

GATEWAY CLASSES
Empowering learners, transforming futures

$$B_n \sinh n\pi = \begin{cases} \dfrac{-8}{\pi n(n^2-4)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases} \qquad n \neq 2$$

For $n = 2$,

$$B_2 \sinh 2\pi = \frac{2}{\pi} \int_0^{\pi} \sin^2 x \sin 2x \, dx$$

$\sin^2 x = 1 - \cos 2x$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\sin 2x - \frac{1}{2} \sin 4x \right) dx$$

$$= \frac{1}{\pi} \left(-\frac{\cos 2x}{2} + \frac{1}{8} \cos 4x \right)_0^{\pi}$$

$$= 0$$

$$\boxed{B_2 = 0}$$

From the eq 11 :

$$u(x,y) = \frac{-8}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{\sin nx \sinh n(\pi - y)}{n(n^2-4) \sinh n\pi}$$

or

$$u(x,y) = \frac{-8}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2m-1)x \sinh (2m-1)(\pi - y)}{(2m-1)\{(2m-1)^2 - 4\} \sinh (2m-1)\pi}$$

↓ For odd we put $\boxed{n = 2m-1}$

Ans

Download Our App from Google PlayStore
Helpline Number : 7455961284
PDF Notes
PYQs Solution
Recorded Lectures