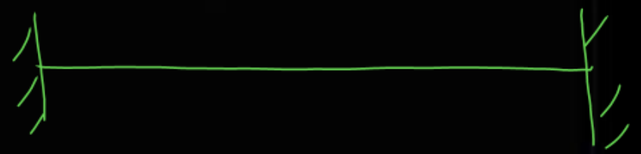
ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (Part – 3)
- PYQ
- DPP



Q.1 : A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially ($t = 0$) at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity ($t = 0$) $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t. (MTU-2011)

The equation of String

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——} \quad (1)$$

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = l$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $y = 0$ IC-1

At $t = 0$, $\frac{dy}{dt} = \lambda x(l - x)$ IC-2

Let $y = XT$ —— (2)

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2} = XT''$$

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2} = TX''$$

Put these values in (1)

$$XT'' = c^2 TX''$$

$$\frac{XT''}{XT} = c^2\frac{TX''}{XT}$$

$$\frac{T''}{T} = c^2\frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2} \times \frac{T''}{T} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad — (3)$$

$$\frac{1}{c^2} \times \frac{T''}{T} = K$$

$$T'' = c^2 KT$$

$$T'' - c^2 KT = 0$$

$$D^2T - c^2 KT = 0$$

$$(D^2 - c^2K)T = 0 \quad — (4)$$

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CASE–1 When $K = 0$

$$D^2X = 0 \qquad D^2T = 0$$

$$D^2 = 0 \qquad D^2 = 0$$

$$D = 0, 0 \qquad D = 0, 0$$

$$CF = C_1 + C_2x \qquad CF = C_3 + C_4t$$

$$PI = 0 \qquad PI = 0$$

$$X = C_1 + C_2x \qquad T = C_3 + C_4t$$

Put X and T in (2)

$$\boxed{y = (C_1 + C_2x)(C_3 + C_4t)} \quad \text{— (5)}$$

CASE –2 : When $K = p^2$

$$(D^2 - p^2)X = 0 \qquad (D^2 - c^2p^2)T = 0$$

$$D^2 - p^2 = 0 \qquad D^2 - c^2p^2 = 0$$

$$D = \pm p \qquad D = \pm cp$$

$$CF = C_1e^{px} + C_2e^{-px} \qquad CF = C_3e^{cpt} + C_4e^{-cpt}$$

$$PI = 0 \qquad PI = 0$$

$$X = C_1e^{px} + C_2e^{-px} \qquad T = C_3e^{cpt} + C_4e^{-cpt}$$

Put X and T in (2)

$$\boxed{y = (C_1e^{px} + C_2e^{-px})(C_3e^{cpt} + C_4e^{-cpt})}$$

— (6)

CASE-3: When $k = -p^2$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D = \sqrt{-p^2}$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$X = C_1 \cos px + C_2 \sin px$$

$$(D^2 + c^2 p^2)T = 0$$

$$D^2 + c^2 p^2 = 0$$

$$D = \sqrt{-c^2 p^2}$$

$$D = \pm icp$$

$$D = 0 \pm icp$$

$$T = C_3 \cos cpt + C_4 \sin cpt$$

Put X and T in (2)

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (7)}$$

Most suitable solution from solutions (5), (6) and (7)

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (8)}$$

Apply Bc-1

$$0 = C_1(C_3\cos cpt + C_4\sin cpt)$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$y = C_2\sin px(C_3\cos cpt + C_4\sin cpt) \quad \text{— (9)}$$

Apply Bc-2

$$0 = C_2\sin pl(C_3\cos cpt + C_4\sin cpt)$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Updated solution

$$y = C_2\sin\frac{n\pi}{l}x\left(C_3\cos\frac{n\pi ct}{l} + C_4\sin\frac{n\pi ct}{l}\right) \quad \text{— (10)}$$

Apply Ic-1

$$0 = C_2 \sin\frac{n\pi x}{l} \times C_3$$

$$\Rightarrow C_3 = 0$$

Updated Solution

$$y = C_2 \sin\frac{n\pi x}{l} \times C_4 \sin\frac{n\pi ct}{l} \quad \text{(11)}$$

Most General solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l} \quad \text{(12)}$$

Where $b_n = C_2 C_4$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times \cos\frac{n\pi ct}{l} \times \frac{n\pi c}{l}$$

Apply IC-2

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times \frac{n\pi c}{l}$$

$$\lambda x(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{l}$$

$$\lambda n(l-n) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin\frac{n\pi n}{l}$$

Half range sine series

$$f(n) = \sum b_n \sin\frac{n\pi n}{l}$$

Where

$$b_n = \frac{2}{l}\int_0^l f(n) \sin\frac{n\pi n}{l}\, dn$$

$$\frac{\pi c}{l} n b_n = \frac{2}{l}\int_0^l \lambda n(l-n) \sin\frac{n\pi n}{l}\, dn$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\int_0^l (ln - n^2) \sin\frac{n\pi n}{l}\, dn$$

I II

$$\int uv\, dn = uv_1 - u'v_2 + u''v_3 - \cdots$$

Formula ↗

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[(lx - x^2)\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (l-2x)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + (-2)\frac{\left(\cos\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[0 - 0 - \frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} - \left(0 - 0 - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[-\frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} + \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$\frac{\pi c}{\ell} n b_n = \frac{2\lambda}{\ell \times \left(\frac{n\pi}{\ell}\right)^3}\left[2 - 2\cos n\pi\right]$$

$$b_n = \frac{2\lambda \times \ell^3}{\pi c n \times n^3\pi^3} \times 2\left(1 - \cos n\pi\right)$$

$$b_n = \frac{4\lambda\ell^3}{c n^4 \pi^4}\left[1 - (-1)^n\right]$$

From (11)

$$y = \sum_{n=1}^{\infty} \frac{4\lambda\ell^3}{c n^4\pi^4}\left[1 - (-1)^n\right] \sin\frac{n\pi x}{\ell} \sin\frac{n\pi c t}{\ell}$$

$$y(x,t) = \frac{4\lambda l^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left[1-(-1)^n\right] \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

If n is even

$$y(x,t) = 0$$

If n is odd

$$y(x,t) = \frac{8\lambda l^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

DPP- 5

Topic : Solution of one Dimensional Wave Equation

Q.1. Show how the wave equation $c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ Can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between $x = 0$ and $x = l$ are given by $y = f(x)$ and $\frac{\partial y}{\partial t} = g(x)$, determine the constants in the series solution. . (UPTU 2014)

Q.2 A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, μ is a constant and then released. Find the displacement $y(x, t)$ of any point x of the string at any time $t > 0$. (M.T.U. 2015)

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