ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Two Dimensional Heat Equation (Laplace Equation) Part-2
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation 3L
2. Solution of one Dimensional Heat Equation 3L
3. Solution of two Dimensional heat Equation 2L

Method of separation of variables 2L

Two Dimensional Heat Equation (Laplace Equation)

(i) Two Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(ii) Two Dimensional Heat Equation in steady state (Laplace Equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

known as Laplace equation

Solution of Laplace equation in two dimension by using Separation of Variables Method.

GATEWAY CLASSES

(1) $$u(x,y) = (C_1 + C_2 x)(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (5)}$$

(2) $$u(x,y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad \text{— (6)}$$

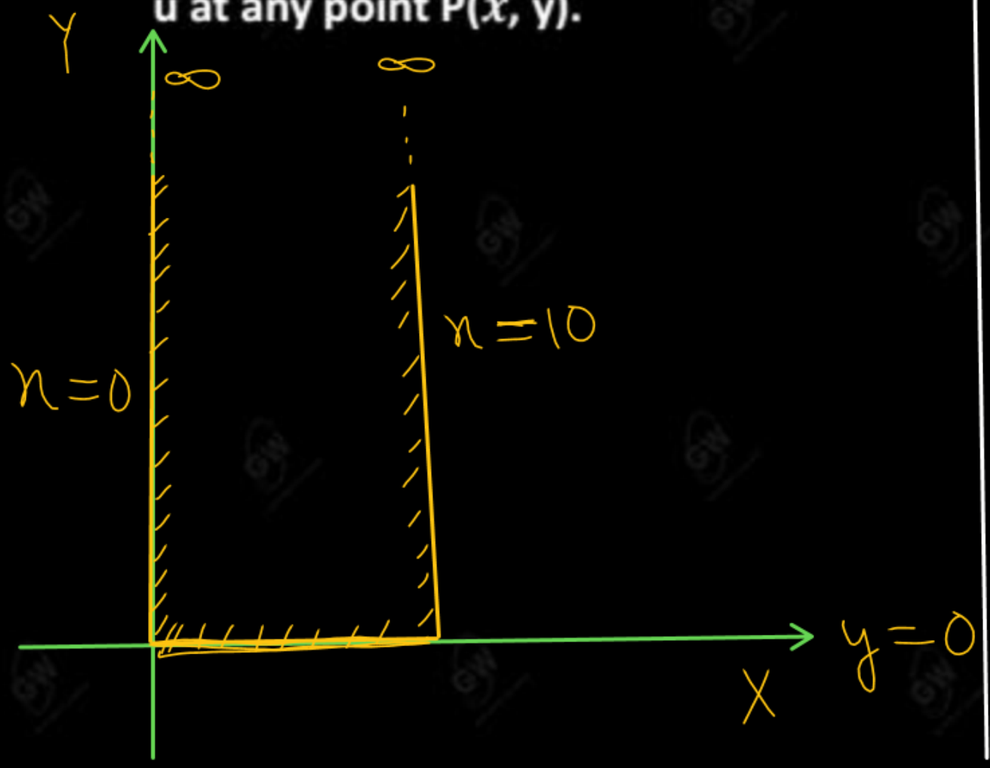
(3) $$u(x,y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (7)}$$

Q.1 A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. IF the temperature along the short edge y = 0 is given by

And $u(x, y) = \begin{cases} 20x, & 0 < x \leq 5 \\ 20(10 - x), & 5 < x < 10 \end{cases}$

And the two long edges x = 0 and x = 10 as well as other short edge are kept at 0°C. Find the temperature u at any point P(x, y).

(A.K.T.U. 2018)



Let $u(x, y)$ be the steady state temp. given by Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

Under the Boundary conditions

$u(0,y) = 0$ BC-1

$u(10,y) = 0$ BC-2

$\lim_{y\to\infty} u(x,y) = 0$ BC-3

$$u(x,0) = \begin{cases} 20x & 0<x\leq 5 \\ 20(10-x) & 5<x<10 \end{cases}$$ BC-4

Solution of eqn (1)

$$u(x,y) = (c_1\cos px + c_2\sin px)(c_3e^{py} + c_4e^{-py})$$ — (7)

Apply BC-1

$$0 = c_1(c_3e^{py} + c_4e^{-py})$$

$$\Rightarrow c_1 = 0$$

Updated solution

$$u(x,y) = c_2\sin px(c_3e^{py} + c_4e^{-py})$$ — (8)

Apply BC-2

$$0 = c_2\sin 10p(c_3e^{py} + c_4e^{-py})$$

$$\Rightarrow \sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

Updated solution

$$u(x,y) = C_2 \sin\frac{n\pi x}{10}\left(C_3 e^{\frac{n\pi y}{10}} + C_4 e^{-\frac{n\pi y}{10}}\right)$$

— (9)

Apply BC-3

$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} C_2 \sin\frac{n\pi x}{10}\left(C_3 e^{\frac{n\pi y}{10}} + C_4 e^{-\frac{n\pi y}{10}}\right)$$

$$0 = C_2 \sin\frac{n\pi x}{10}\left[\lim_{y\to\infty} C_3 e^{\frac{n\pi y}{10}} + \lim_{y\to\infty} C_4 e^{-\frac{n\pi y}{10}}\right]$$

$$0 = C_2 \sin\frac{n\pi x}{10}\left[\lim_{y\to\infty} C_3 e^{\frac{n\pi y}{10}}\right]$$

$$\Rightarrow C_3 = 0$$

Updated solution

$$u(x,y) = C_2 \sin\frac{n\pi x}{10} \times C_4 e^{-\frac{n\pi y}{10}}$$

— (10)

$$u(x,y) = c_2 c_4 \sin\frac{n\pi x}{10} \times e^{-\frac{n\pi y}{10}} \quad \text{— (11)}$$

Most general solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \times e^{-\frac{n\pi y}{10}}$$

— (12)

Apply BC-4

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Apply Half Range sine series

$$b_n = \frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{10}\int_0^{10} u(x,0)\sin\frac{n\pi x}{10}\,dx$$

$$b_n = \frac{1}{5}\left[\int_0^5 u(x,0)\sin\frac{n\pi x}{10}dx + \int_5^{10} u(x,0)\sin\frac{n\pi x}{10}dx\right]$$

$$b_n = \frac{1}{5}\int_0^5 20x\sin\frac{n\pi x}{10}dx + \frac{1}{5}\int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}dx$$

$$b_n = \frac{20}{5}\int_0^5 x \sin\frac{n\pi x}{10}\,dx + \frac{20}{5}\int_5^{10} (10-x)\sin\frac{n\pi x}{10}\,dx$$

$$b_n = 4\left[x\frac{\left(-\cos\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} - 1x\frac{\left(-\sin\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_0^5 + 4\left[(10-x)\frac{\left(-\cos\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} - (-1)\frac{\left(-\sin\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_5^{10}$$

$$b_n = 4\left[\frac{-5\cos\frac{n\pi}{2}}{\frac{n\pi}{10}} + \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{10}\right)^2}\right] + 4\left[0+0-\left(\frac{-5\cos\frac{n\pi}{2}}{\frac{n\pi}{10}} - \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]$$

$$b_n = 4\left(\frac{-50}{n\pi}\cos n\pi + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + 4\left(\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$b_n = \frac{-200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{\pi^2 n^2}\sin\frac{n\pi}{2} + \frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

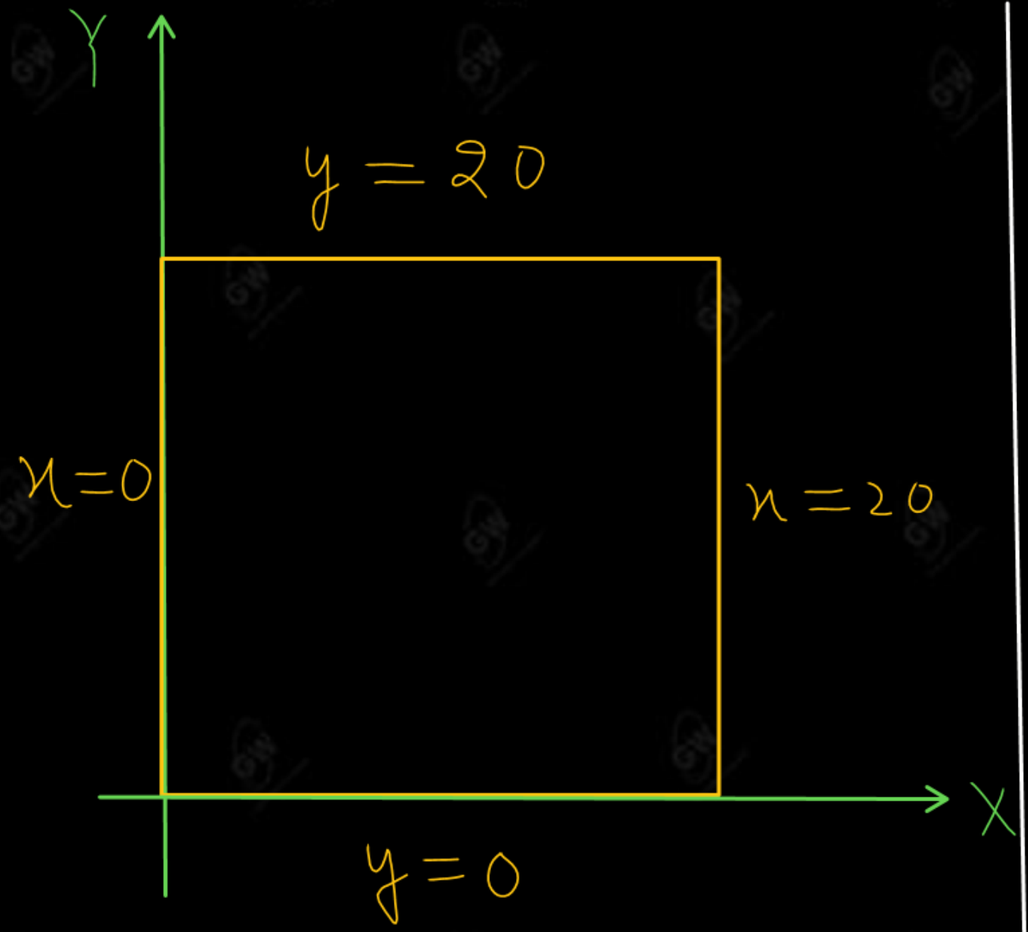
$$b_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Put b_n in (12)

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

$$\boxed{u(x,y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}}$$

Q.2:- A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while other three edges are kept 0°C . Find the steady state temperature in the plate. *(A.K.T.U. 2014, 2017)*



Let $u(x, y)$ be the steady state temp given by Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

Under the Boundary conditions

$u(0, y) = 0$ BC-1

$u(x, 0) = 0$ BC-2

$u(20, y) = 0$ BC-3

$u(x, 20) = x(20 - x)$ BC-4

Solution of eqn (1)

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \text{— (7)}$$

Apply BC-1

$$0 = C_1 (C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$u(x, y) = C_2 \sin px (C_3 e^{py} + C_4 e^{-py}) \quad \text{— (8)}$$

Apply BC-2

$$0 = C_2 \sin px (C_3 + C_4)$$

$$\Rightarrow C_3 + C_4 = 0$$

$$C_4 = -C_3$$

Updated solution

$$u(x, y) = C_2 \sin px (C_3 e^{py} - C_3 e^{-py})$$

$$u(x,y) = C_2 C_3 \sin px \left(e^{py} - e^{-py}\right) \quad \text{— (9)}$$

Apply BC-3

$$0 = C_2 C_3 \sin 20p \left(e^{py} - e^{-py}\right)$$

$$\Rightarrow \sin 20p = 0$$

$$\sin 20p = \sin n\pi$$

$$20p = n\pi$$

$$p = \frac{n\pi}{20}$$

Updated solution

$$u(x,y) = C_2 C_3 \sin \frac{n\pi}{20} x \left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right) \quad \text{— (10)}$$

Most General solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right) \quad \text{— (11)}$$

Apply BC-4

$$x(20-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \left(e^{n\pi} - e^{-n\pi}\right)$$

$$(20x - x^2) = \sum_{n=1}^{\infty} b_n' \sin\frac{n\pi x}{20}$$

Apply Half range sine-series

$$b_n' = \frac{2}{l}\int_0^l u(x,20)\sin\frac{n\pi x}{l}\,dx$$

$$b_n' = \frac{2}{20}\int_0^{20} (20x - x^2)\sin\frac{n\pi x}{20}\,dx$$

I II

$$b_n' = \frac{1}{10}\left[(20x - x^2)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - (20-2x)\left(\frac{-\sin\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^2}\right) + (-2)\left(\frac{+\cos\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^3}\right)\right]_0^{20}$$

$$b_n' = \frac{1}{10}\left[0 + 0 - \frac{2\cos n\pi}{\left(\frac{n\pi}{20}\right)^3} - \left(0 + 0 - \frac{2}{\left(\frac{n\pi}{20}\right)^3}\right)\right]$$

$$b_n' = \frac{1}{10}\left[-\frac{2\times 8000}{n^3\pi^3}\cos n\pi + \frac{2\times 8000}{n^3\pi^3}\right]$$

$$b_n' = \frac{2\times 8000}{10\times n^3\pi^3}\left[1 - \cos n\pi\right]$$

$$b_n' = \frac{1600}{n^3\pi^3}\left[1 - (-1)^n\right]$$

When n is even

$$b_n' = 0$$

When n is odd

$$b_n' = \frac{3200}{n^3\pi^3}$$

$$b_n\left(e^{n\pi} - e^{-n\pi}\right) = \frac{3200}{n^3\pi^3}$$

$$2b_n\left(\frac{e^{n\pi} - e^{-n\pi}}{2}\right) = \frac{3200}{n^3\pi^3}$$

$$2 b_n \sinh n\pi = \frac{3200}{n^3\pi^3}$$

$$b_n = \frac{1 \times 3200}{2\sinh n\pi \times n^3\pi^3}$$

Put b_n in (11)

$$u(x,y) = \sum_{n=1}^{\infty} \frac{3200}{2\sinh n\pi \times n^3\pi^3} \sin\frac{n\pi x}{20}\left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right)$$

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{20} \times 2}{2n^3\sinh n\pi}\left(\frac{e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}}{2}\right)$$

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{20}}{n^3}\frac{\sinh\left(\frac{n\pi y}{20}\right)}{\sinh n\pi}$$

Where n is odd

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\frac{(2n-1)\pi x}{20} \times \sinh\frac{(2n-1)\pi y_n}{20}}{(2n-1)^3\sinh(2n-1)\pi}$$

Engineering Mathematics by Gulshan Sir

DPP- 10

Topic : Two Dimensional Heat Equation (Laplace Equation)

Part-2

Q.1. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle in the xy – plane with $u(x, 0) = 0$, $u(x, b) = 0$, $u(0,y) = 0$ and $u(a, y) = f(y)$ parallel to y-axis. (U.K.T.U. 2016)

Q.2 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0, y) = 0$, $u(a, y) = 0$, $u(x, 0) = 0$ and $u(x, b) = x$. *[U.P.T.U. 2015]*

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