ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Fourier cosine transform***
- PYQ
- DPP

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or f(p) and defined as

$$F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx$$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipx}dx$$

Fourier sine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$ or $f_s(p)$ and defined as

$$F\{F(x)\} = f_s(p) = \int_0^\infty F(x) sinpx dx$$

Where p is a parameter and $p > 0$

Inverse Fourier sine transform

If $F_s\{F(x)\} = f_s(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier sine transform and defined as

$$F_s^{-1}\{f_s(p)\} = F(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} f_s(p)\, sinpx dp$$

Fourier cosine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier cosine transform of $F(x)$ is denoted by $F_c\{F(x)\}$ or $f_c(p)$ and defined as

$$F_c\{F(x)\} = f_c(p) = \int_0^\infty F(x)\cos px\,dx$$

Where p is a parameter and $p > 0$

Inverse Fourier cosine transform

If $F_c\{F(x)\} = f_c(p)$ is the Fourier cosine transform of $F(x)$, then $F(x)$ is called inverse Fourier cosine transform and defined as

$$F_c^{-1}\{f_c(p)\} = F(x) = \frac{2}{\pi}\int_0^\infty f_c(p)\cos px\,dp$$

Some Important Points and Formulae

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Two Important results

(i) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + c$

(ii) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + c$

Trigonometric Identities

(1) $2 \sin A \cos A = \sin 2A$

(2) $2 \sin A \cos B = \sin (A+B) + \sin (A - B)$

(3) $2 \cos A \sin B = \sin (A+B) - \sin (A - B)$

(4) $2 \cos A \cos B = \cos (A+B) + \cos (A - B)$

(5) $2 \sin A \sin B = \cos (A-B) - \cos (A + B)$

Q.1. *Find the Fourier cosine transform of* $F(x) = e^{-2x} + 4e^{-3x}$

$$F(x) = e^{-2x} + 4e^{-3x}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x)\cos px\,dx$$

$$f_c(p) = \int_0^\infty (e^{-2x} + 4e^{-3x})\cos px\,dx$$

$$f_c(p) = \int_0^\infty e^{-2x}\cos px\,dx + 4\int_0^\infty e^{-3x}\cos px\,dx$$

$$f_c(p) = \left[\frac{e^{-2x}}{(-2)^2 + p^2}(-2\cos px + p\sin px)\right]_0^\infty$$

$$+ 4\left[\frac{e^{-3x}}{(-3)^2 + p^2}(-3\cos px + p\sin px)\right]_0^\infty$$

$$f_c(p) = \left[0 - \left\{\frac{1}{4+p^2}(-2)\right\}\right]$$

$$+ 4\left[0 - \left\{\frac{1}{9+p^2}(-3)\right\}\right]$$

$$f_c(p) = \frac{2}{p^2+4} + \frac{12}{p^2+9}$$

Q.2 . Find Fourier cosine transform of $F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

$$F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^a F(x) \cos px \, dx + \int_a^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^a \cos x \cos px \, dx + 0$$

$$f_c(p) = \frac{1}{2} \int_0^a 2 \cos x \cos px \, dx$$

$$f_c(p) = \frac{1}{2} \int_0^a \{\cos(x+px) + \cos(x-px)\} dx$$

$$f_c(p) = \frac{1}{2} \int_0^a \{\cos(1+p)x + \cos(1-p)x\} dx$$

$$f_c(c) = \frac{1}{2} \left[\frac{\sin(1+p)x}{1+p} + \frac{\sin(1-p)x}{1-p} \right]_0^a$$

$$f_c(p) = \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p} - (0+0)\right]$$

$$\boxed{f_c(p) = \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p}\right]}$$

Q.3 . Obtain Fourier cosine transform of $F(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

$$F(x) = \begin{cases} x & , \quad 0 < x < 1 \\ 2 - x & , \quad 1 < x < 2 \\ 0 & , \quad x > 2 \end{cases}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^1 F(x) \cos px \, dx + \int_1^2 F(x) \cos px \, dx + \int_2^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^1 \underset{I}{x} \underset{II}{\cos px} \, dx + \int_1^2 \underset{I}{(2-x)} \underset{II}{\cos px} \, dx + 0$$

$$f_c(p) = \left[x\left(\frac{\sin px}{p}\right) - 1\left(\frac{-\cos px}{p^2}\right) \right]_0^1$$

$$+ \left[(2-x)\left(\frac{\sin px}{p}\right) - (-1)\left(\frac{-\cos px}{p^2}\right) \right]_1^2$$

$$f_c(p) = \left[\frac{x \sin px}{p} + \frac{\cos px}{p^2}\right]_0^1 + \left[(2-x)\frac{\sin px}{p} - \frac{\cos px}{p^2}\right]_1^2$$

$$f_c(p) = \frac{\sin p}{p} + \frac{\cos p}{p^2} - \left(0 + \frac{1}{p^2}\right) + 0 - \frac{\cos 2p}{p^2} - \left(\frac{\sin p}{p} - \frac{\cos p}{p^2}\right)$$

$$f_c(p) = \frac{\sin p}{p} + \frac{\cos p}{p^2} - \frac{1}{p^2} - \frac{\cos 2p}{p^2} - \frac{\sin p}{p} + \frac{\cos p}{p^2}$$

$$\boxed{f_c(p) = \frac{2\cos p}{p^2} - \frac{\cos 2p}{p^2} - \frac{1}{p^2}}$$

$$f_c(p) = \frac{2\cos p}{p^2} - \left(\frac{2\cos^2 p - 1}{p^2}\right) - \frac{1}{p^2} \qquad \{\because \cos 2\theta = 2\cos^2\theta - 1\}$$

$$f_c(p) = \frac{2\cos p}{p^2} - \frac{2\cos^2 p}{p^2} + \frac{1}{p^2} - \frac{1}{p^2}$$

$$f_c(p) = \frac{2\cos p - 2\cos^2 p}{p^2}$$

$$\boxed{f_c(p) = \frac{2\cos p(1-\cos p)}{p^2}}$$

Q.4. Find Fourier cosine transform of $\frac{1}{1+x^2}$ and hence find Fourier sine transform of $\frac{x}{1+x^2}$. (AKTU-2018, 2017)

$$F(x) = \frac{1}{1+x^2}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$$

Let $f_c(p) = I$

$$I = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx \quad \text{——} (1)$$

Differentiate w.r.t p

$$\frac{dI}{dp} = \int_0^\infty \frac{1}{1+x^2} \frac{d}{dp}(\cos px) \, dx$$

$$\frac{dI}{dp} = \int_0^\infty \frac{1}{1+x^2} \times x (-\sin px) \, dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{x \sin px}{1+x^2} \, dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{x^2 \sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{(1+x^2-1)\sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{(1+x^2)\sin px - \sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\left[\int_0^\infty \left\{\frac{(1+x^2)\sin px}{x(1+x^2)} - \frac{\sin px}{x(1+x^2)}\right\}dx\right.$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{\sin px}{x}\,dx + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx \quad \text{— (2)}$$

Differentiate w.r.t p

$$\frac{d^2I}{dp^2} = 0 + \int_0^\infty \frac{x\cos px}{x(1+x^2)}\,dx$$

$$\frac{d^2I}{dp^2} = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$$

$$\frac{d^2I}{dp^2} = I$$

$$\frac{d^2I}{dp^2} - I = 0$$

solution of above Differential equation is

$$I = c_1 e^{p} + c_2 e^{-p}$$

$$I = c_1 e^{p} + c_2 e^{-p} \quad \text{— (3)}$$

diff w.r.t p

$$\frac{dI}{dp} = c_1 e^{p} - c_2 e^{-p} \quad \text{— (4)}$$

When $p = 0$

From (3)

$$I = c_1 + c_2 \quad \text{— (5)}$$

From (4)

$$\frac{dI}{dp} = c_1 - c_2 \quad \text{— (6)}$$

From (1)

$$I = \int_0^\infty \frac{dx}{1+x^2}$$

$$I = \left(\tan^{-1} x\right)_0^\infty$$

$$I = \tan^{-1}\infty - \tan^{-1} 0$$

$$I = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{2}$$

From (2)

$$\frac{dI}{dp} = -\frac{\pi}{2}$$

Put I and $\frac{dI}{dp}$ in (5) and (6)

$$c_1 + c_2 = \frac{\pi}{2} \quad \text{— (7)}$$

$$c_1 - c_2 = -\frac{\pi}{2} \quad \text{— (8)}$$

Solve (7) and (8)

$$c_1 + c_2 = \frac{\pi}{2}$$
$$c_1 - c_2 = -\frac{\pi}{2}$$

$$2c_1 = 0$$

$$\boxed{c_1 = 0}$$

$$\boxed{c_2 = \frac{\pi}{2}}$$

Put c_1 and c_2 in (3)

$$I = \frac{\pi}{2} e^{-p}$$

$$\int_0^\infty \frac{1}{1+x^2}\cos px\,dx = \frac{\pi}{2}e^{-p}$$

$$\int_0^\infty \frac{x}{1+x^2}\sin px\,dx = \frac{\pi}{2}e^{-p}$$

diff wrt p

$$\int_0^\infty \frac{1}{1+x^2}\frac{d}{dp}(\cos px)\,dx = \frac{\pi}{2}\frac{d}{dp}(e^{-p})$$

$$\int_0^\infty -\frac{x\sin px}{1+x^2}\,dx = -\frac{\pi}{2}e^{-p}$$

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Topic : *Fourier cosine transform*

Q.1. *Find Fourier cosine transform* $F(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1 \\ 0, & x > 1 \end{cases}$

Q. 2. *Find the Fourier cosine transform of* e^{-x^2}

Q.3. *Find Fourier cosine transform of* $F(x) = 2e^{-5x} + 2e^{-2x}$

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