ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (Part – 2)
- PYQ
- DPP

Q.1 : Find the deflection $y(x,t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x, \quad 0 \le x \le 1$$

(G.B.T.U. 2013)

The equation of string

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

length of string

$l = 1$ unit

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = 1$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $\frac{\partial y}{\partial t} = 0$ IC-1

At $t = 0$, $y = \sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x$ IC-2

Most suitable solution

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad -(8)$$

Apply BC-1

$$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$$

$$C_1 = 0$$

Updated solution

$$y = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt) \quad -(9)$$

Apply BC-2

$$0 = C_2 \sin p (C_3 \cos cpt + C_4 \sin cpt)$$

$$\Rightarrow \sin p = 0$$

$$\sin p = \sin n\pi$$

$$p = n\pi$$

Updated solution

$$y = C_2 \sin n\pi x \left(C_3 \cos \frac{n\pi c}{} t + C_4 \sin \frac{n\pi c}{} t \right) \quad \text{—(10)}$$

$$\frac{\partial y}{\partial t} = C_2 \sin n\pi x \left(-C_3 \times n\pi c \sin n\pi c t + C_4 \times n\pi c \cos n\pi c t \right)$$

Apply I c-1

$$0 = C_2 \sin n\pi x \left(0 + n\pi c C_4 \right)$$

$$0 = C_2 C_4 n\pi c \sin n\pi x$$

$$\Rightarrow C_4 = 0$$

Updated solution

$$y = C_2 \sin n\pi x \times C_3 \cos n\pi c t$$

$$y = (C_2 C_4) \sin n\pi x \cos n\pi c t$$

Most general solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cos n\pi c t \quad \text{— (11)}$$

Apply I.C-2

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x = b_1 \sin \pi x + b_2 \sin 2\pi x + b_3 \sin 3\pi x + b_4 \sin 4\pi x + b_5 \sin 5\pi x + \cdots$$

By comparison

$b_1 = 1$

$b_2 = 0$

$b_3 = \frac{1}{3}$

$b_4 = 0$

$b_5 = \frac{1}{5}$

From (1)

$$y(x,t) = b_1 \sin \pi x \cos \pi c t + b_2 \sin 2\pi x \cos 2\pi c t + b_3 \sin 3\pi x \cos 3\pi c t + b_4 \sin 4\pi \cos 4\pi c t + b_5 \sin 5\pi x \cos 5\pi c t$$

$$y(x,t) = \sin \pi x \cos \pi c t + \frac{1}{3} \sin 3\pi x \cos 3\pi c t + \frac{1}{5} \sin 5\pi x \cos 5\pi c t$$

Q.2. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$.

Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x, t) = A \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}$$

(AKTU-2018, 2013; UKTU-2011, 2012,2022)

The equation of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

length of string

$l = l$

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = l$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $\frac{\partial y}{\partial t} = 0$ IC-1

At $t = 0$, $y = A \sin \frac{\pi x}{l}$ IC-2

Most suitable solution

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (8)}$$

Apply Bc-1

$$0 = C_1 (C_3 \cos cpt + C_4 \sin cpt)$$

$$\Rightarrow C_1 = 0$$

Updated solution

$$y = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt) \quad \text{— (9)}$$

Apply Bc-2

$$0 = C_2 \sin pl (C_3 \cos cpt + C_4 \sin cpt)$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Updated solution

$$y = C_2 \sin\frac{n\pi}{l}x\left(C_3\cos\frac{n\pi ct}{l} + C_4\sin\frac{n\pi ct}{l}\right) \quad \text{—— (10)}$$

$$\frac{dy}{dt} = C_2\sin\frac{n\pi x}{l}\left(-C_3 x\frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + C_4 x\frac{n\pi c}{l}\cos\frac{n\pi ct}{l}\right)$$

Apply Ic-1

$$0 = C_2\sin\frac{n\pi x}{l} \times C_4 x\frac{n\pi c}{l}$$

$$\Rightarrow C_4 = 0$$

Updated solution

$$y = C_2 \sin\frac{n\pi x}{l} \times C_3 \cos\frac{n\pi ct}{l}$$

$$y = (C_2 C_4) \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

Most General solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

└ (11)

Apply IC-2

$$A \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$A \sin\frac{\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \cdots$$

on comparson

$$b_1 = A$$

From (11)

$$y(x,t) = b_1 \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l} + - - -$$

$$y(x,t) = A \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l}$$

Hence proved

Engineering Mathematics by Gulshan Sir

DPP- 4

Topic : Solution of one Dimensional Wave Equation (Part -2)

Q.1. Find the deflection of the vibrating string which is fixed at the ends $x = 0$ and $x = 2$ and the motion is started by displacing the string into the form $sin^3\left(\frac{\pi x}{2}\right)$ and releasing it with zero initial velocity at $t = 0$ *(U.P.T.U. 2014)*

Q.2 A string *is* stretched and fastened to two points distance l apart. Find the displacement $y(x, t)$ at any point at a distance x from one end at time t given that:

$$siny(x, 0) = A\, sin\left(\frac{2\pi x}{l}\right)$$

(M.T.U. 2013)

Q.3 A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0\, sin^3\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.

(GBTU-2011,2020)

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Thank You

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